A STUDY ON L-FUZZY NORMAL SUB ℓ-GROUP

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ABSTRACT

This paper contains some definitions and results of L-fuzzy normal sub ℓ-group and its generalized characteristics.

KEYWORDS

Fuzzy set, L -fuzzy set, L-fuzzy sub ℓ --group, L-fuzzy normal sub ℓ -group .

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1.INTRODUCTION

L. A. Zadeh[11] introduced the notion of fuzzy subset of a set S as a function from X into I = [0, 1]. Rosenfeld[2] applied this concept in group theory and semi group theory, and developed the theory of fuzzy subgroups and fuzzy subsemigroupoids respectively J.A.Goguen [6] replaced the valuations set [0, 1],by means of a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L-fuzzy sets.. In fact it seems in order to obtain a complete analogy of crisp mathematics in terms of fuzzy mathematics, it is necessary to replace the valuation set by a system having more rich algebraic structure. These concepts ℓ-groups play a major role in mathematics and fuzzy mathematics. G.S.V Satya Saibaba [9] introduced the concept of L- fuzzy sub ℓ-group and L-fuzzy ℓ-ideal of ℓ-group. In this paper, we initiate the study of L-fuzzy normal sub ℓ-groups .

2.PRELIMINARIES

This section contains some definitions and results to be used in the sequel.

2.1. Definition [5,6,7]

A lattice ordered group (ℓ -group) is a system G= (G, *, $\leq$) where

i (G, *) is a group
ii (G , $\leq$) is a lattice
iii the inclusion is invariant under all translations
$x \rightarrow a + x + b$ i.e. $x \leq y \Rightarrow a + x + b \leq a + y + b$, for all $a, b \in G$.

2.2 .Definition [11]

Let X be a non-empty set. A fuzzy subset A of X is a function
$A : X \rightarrow [\, 0, 1\,]$.

2.3. Definition [1,2]

An L-fuzzy subset A of G is called an L-fuzzy subgroup (ALFS) of G if for every x,y $\in$ G,

i $\quad A(xy) \geq A(x) \vee A(y)$
ii $\quad A(x^{-1}) = A(x)$.

2.4. Definition [9,10]

An L-fuzzy subset A of G is said to be an L-fuzzy sub ℓ - group(LFSℓ) of G if for any x , y $\in$ G

i. $\quad A(xy) \geq A(x) \vee A(y)$
ii. $\quad A(x^{-1}) = A(x)$
iii. $\quad A(x \vee y) \geq A(x) \vee A(y)$
iv. $\quad A(x \wedge y) \geq A(x) \vee A(y)$.

2.5. Definition [4]

Let G and G′ be any two groups. Then the function f: G → G′ is said to be a homomorphism if f (xy) = f (x) f (y) for all x, y in G.

2.6.Definition[3]

Let G and G′ be any two groups (not necessarily commutative). Then the function f: G → G′ is said to be an anti-homomorphism if f (xy) = f (y) f (x) for all x, y in G.

Remark: A homomorphism may or may not be an anti-homomorphism

2.7 .Definition [8,10]

A subℓ-group H of an ℓ - group G is called a normal subℓ-group of G if for all x in G and h in H we have $xhx^{-1} \in H$.

2.8.Definition[8,10]

An L-fuzzy sub ℓ **-**group A of G is called an L-fuzzy normal sub ℓ **-**group (LFNSℓG) of G if for every x, y$\in$G, $A(xyx^{-1}) \geq A(y)$.

3. PROPERTIES OF AN L-FUZZY NORMAL SUB ℓ-GROUP

In this section, we discuss properties of an L-fuzzy normal sub ℓ-group

3.1.Theorem

Let G be an ℓ-group and A be an L-fuzzy sub ℓ-group of G, then the following conditions are equivalent.

i. A is an L-fuzzy normal sub ℓ-group of G.
ii. $A(xyx^{-1}) = A(y)$, for all $x, y \in G$.
iii. $A(xy) = A(yx)$, for all $x, y \in G$.
iv.

Proof:

$i \Rightarrow ii$.

Let A is an L-fuzzy normal sub ℓ-group of G.

Then $A(xyx^{-1}) \geq A(y)$ for all $x, y \in G$. By taking advantage of the arbitrary property of x, we have,

$$A(x^{-1}y(x^{-1})^{-1}) \geq A(y).$$

Now,

$$A(y) = A(x^{-1}(xyx^{-1})(x^{-1})^{-1})$$
$$= A(xyx^{-1})$$
$$\geq A(y).$$

Hence,

$$A(xyx^{-1}) = A(y) \text{ for all } x, y \in G.$$

$ii \Rightarrow iii$.

Let $A(xyx^{-1}) = A(y)$, for all $x, y \in G$.

Taking yx instead of y, we get,

$$A(xy) = A(yx), \text{ for all } x, y \in G.$$

$iii \Rightarrow i$.

Let $A(xy) = A(yx)$, for all $x, y \in G$.

$$A(xyx^{-1}) = A(yxx^{-1}) = A(y) \geq A(y).$$

Hence, A is an L-fuzzy normal sub ℓ-group of G.

3.2 .Theorem

Let A be an L-fuzzy subset of an ℓ-group G. If A(e) = 1 and $A(xy^{-1}) \geq A(x) \wedge A(y)$, $A(x \vee y) \geq A(x) \wedge A(y)$, $A(x \wedge y) \geq A(x) \wedge A(y)$ and $A(xy) = A(yx)$, for all x and y in G, then A is an L-fuzzy normal sub ℓ-group of a group G, where e is the identity element of G.

Proof:

Let e be identity element of G and x and y in G.

Let A(e) = 1 and $A(xy^{-1}) \geq A(x) \wedge A(y)$, for all x and y in G.

$$
\begin{aligned}
\text{Now,} \quad A(x^{-1}) &= A(ex^{-1}) \\
&\geq A(e) \wedge A(x) \\
&\geq 1 \wedge A(x) \\
&= A(x)
\end{aligned}
$$

Therefore, $A(x^{-1}) \geq A(x)$, for all x in G.

Hence, $A((x^{-1})^{-1}) \geq A(x^{-1})$ and $A(x) \geq A(x^{-1})$.

Therefore, $A(x^{-1}) = A(x)$, for all x in G.

Now, replace y by y^{-1}, then

$$
\begin{aligned}
A(xy) &= A(x(y^{-1})^{-1}) \\
&\geq A(x) \wedge A(y^{-1}) \\
&= A(x) \wedge A(y), \text{ for all x and y in G.}
\end{aligned}
$$

$$A(xy) \geq A(x) \wedge A(y), \text{ for all x and y in G.}$$

Also, we have, $A(x \vee y) \geq A(x) \wedge A(y)$, $A(x \wedge y) \geq A(x) \wedge A(y)$.

Hence, A is an L-fuzzy sub ℓ-group of an ℓ-group G.

Since, A(xy) = A(yx) for all x and y in G, A is an L-fuzzy normal sub ℓ-group of an ℓ-group G.

3.3 .Theorem

If A is an L-fuzzy normal sub ℓ-group of an ℓ-group G, then $H = \{x / x \in G: A(x) = 1\}$ is either empty or a normal sub ℓ-group of G.

Proof

It is clear from theorem 3.2

3.4 .Theorem

If A is an L-fuzzy normal sub ℓ-group of an ℓ-group G, then $H = \{x \in G : A(x) = A(e)\}$ is either empty or a normal sub ℓ-group of G, where e is the identity element of G.

Proof

Since , H is a sub ℓ-group of G.

Now, let for any x in G and y in H, $A(xyx^{-1}) = A(y) = A(e)$.

Since A is an LFNSℓG of an ℓ-group G and $y \in H$.

Hence, $xyx^{-1} \in G$ and H is a normal sub ℓ-group of G.

Hence, H is either empty or a normal sub ℓ-group of an ℓ-group G.

3.5 .Theorem

If A and B are two L-fuzzy normal sub ℓ-groups of an ℓ-group G, then their intersection $A \cap B$ is an L-fuzzy normal sub ℓ-group of G.

Proof

Let x and y belong to G.

i.
$$\begin{aligned}(A \cap B)(xy) &= A(xy) \wedge B(xy) \\ &\geq \{ A(x) \wedge A(y) \} \wedge \{ B(x) \wedge B(y) \} \\ &\geq \{ A(x) \wedge B(x) \} \wedge \{ A(y) \wedge B(y) \} \\ &= (A \cap B)(x) \wedge (A \cap B)(y).\end{aligned}$$

Therefore, $(A \cap B)(xy) \geq (A \cap B)(x) \wedge (A \cap B)(y)$, for all x and y in G.

ii.
$$\begin{aligned}(A \cap B)(x^{-1}) &= A(x^{-1}) \wedge B(x^{-1}) \\ &= A(x) \wedge B(x) \\ &= (A \cap B)(x).\end{aligned}$$

Therefore, $(A \cap B)(x^{-1}) = (A \cap B)(x)$, for all x in G.

iii.
$$\begin{aligned}(A \cap B)(x \vee y) &= A(x \vee y) \wedge B(x \vee y) \\ &\geq \{ A(x) \wedge A(y) \} \wedge \{ B(x) \wedge B(y) \} \\ &\geq \{ A(x) \wedge B(x) \} \wedge \{ A(y) \wedge B(y) \} \\ &= (A \cap B)(x) \wedge (A \cap B)(y).\end{aligned}$$

Therefore, $(A \cap B)(x \vee y) \geq (A \cap B)(x) \wedge (A \cap B)(y)$, for all x and y in G.

iv. $(A \cap B)(x \vee y) = A(x \vee y) \wedge B(x \vee y)$

$\geq \{ A(x) \wedge A(y) \} \wedge \{ B(x) \wedge B(y) \}$

$\geq \{ A(x) \wedge B(x) \} \wedge \{ A(y) \wedge B(y) \}$

$= (A \cap B)(x) \wedge (A \cap B)(y)$.

Therefore, $(A \cap B)(x \vee y) \geq (A \cap B)(x) \wedge (A \cap B)(y)$, for all x and y in G.

Hence, $A \cap B$ is an L-fuzzy sub ℓ-group of an ℓ-group G.

Now, $(A \cap B)(xy) = A(xy) \wedge B(xy)$

$= A(yx) \wedge B(yx)$, since A and B are LFNS ℓ G of G.

$= (A \cap B)(yx)$.

$(A \cap B)(xy) = (A \cap B)(yx)$.

Hence, $A \cap B$ is an L-fuzzy normal sub ℓ-group of an ℓ-group G.

Remark

The intersection of a family of L-fuzzy normal sub ℓ-groups of an ℓ-group G is an L-fuzzy normal sub ℓ-group of an ℓ-group G.

3.6 .Theorem

If A is an L-fuzzy normal sub ℓ-group of an ℓ-group G if and only if $A(x) = A(y^{-1}xy)$, for all $x, y \in G$.

Proof

Let x and y be in G. Let A be an L-fuzzy normal sub ℓ-group of an ℓ-group G.

Now, $A(y^{-1}xy) = A(y^{-1}yx)$

$= A(ex)$

$= A(x)$.

Therefore, $A(x) = A(y^{-1}xy)$, for all x and y in G.

Conversely, assume that $A(x) = A(y^{-1}xy)$.

Now, $A(xy) = A(xyxx^{-1})$

$= A(yx)$

Therefore, $A(xy) = A(yx)$, for all x and y in G.

Hence, A is an L-fuzzy normal sub ℓ-group of an ℓ-group G.

3.7 .Theorem

Let A be an L-fuzzy sub ℓ-group of an ℓ-group G with $A(y) < A(x)$, for some x and y in G, then A is an L-fuzzy normal sub ℓ-group of an ℓ-group G.

Proof

Let A be an L-fuzzy sub ℓ-group of an ℓ-group G.

Given $A(y) < A(x)$, for some x and y in G,

$$\begin{aligned} A(xy) &\geq A(x) \wedge A(y), \text{ as A is an LFS}\ell\text{G of G} \\ &= A(y); \text{ and} \\ A(y) &= A(x^{-1}xy) \\ &\geq A(x^{-1}) \wedge A(xy) \\ &\geq A(x) \wedge A(xy), \text{ as A is an LFS}\ell\text{G of G} \\ &= A(xy). \\ A(y) &\geq A(xy) \geq A(y). \end{aligned}$$

Therefore, $A(xy) = A(y)$, for all x and y in G.

and,

$$\begin{aligned} A(yx) &\geq A(y) \wedge A(x), \text{ as A is an LFS}\ell\text{G of G} \\ &= A(y); \text{ and} \\ A(y) &= A(yxx^{-1}) \\ &\geq A(yx) \wedge A(x^{-1}) \\ &\geq A(yx) \wedge A(x), \text{ as A is an LFS}\ell\text{G of G} \\ &= A(yx). \\ A(y) &\geq A(yx) \geq A(y). \end{aligned}$$

Therefore, $A(yx) = A(y)$, for all x and y in G.

Hence, $A(xy) = A(y) = A(yx)$, for all x and y in G.

Hence, $A(xy) = A(yx)$, for all x and y in G.

Hence, A is an L-fuzzy normal sub ℓ-group of an ℓ-group of G.

3.8 .Theorem

Let A be an L-fuzzy sub ℓ-group of an ℓ-group G with $A(y) > A(x)$ for some x and y in G, then A is an L-fuzzy normal sub ℓ-group of an ℓ-group G.

Proof

It is clear from theorem 3.7

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