APPROXIMATE ANALYTICAL SOLUTION OF NON-LINEAR BOUSSINESQ EQUATION FOR THE UNSTEADY GROUND WATER FLOW IN AN UNCONFINED AQUIFER BY HOMOTOPY PERTURBATION TRANSFORM METHOD

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Abstract

For one dimensional homogeneous, isotropic aquifer, without accretion the governing Boussinesq equation under Dupuit assumptions is a nonlinear partial differential equation. In the present paper approximate analytical solution of nonlinear Boussinesq equation is obtained using Homotopy perturbation transform method(HPTM). The solution is compared with the exact solution. The comparison shows that the HPTM is efficient, accurate and reliable. The analysis of two important aquifer parameters namely viz. specific yield and hydraulic conductivity is studied to see the effects on the height of water table. The results resemble well with the physical phenomena.

Keywords

Aquifers, stream-aquifer interaction, Dupuit assumptions, Boussinesq equation, Homotopy perturbation transform method.

1.Introduction

Studies**[9]**have shown that the total amount of surface and ground water is not enough for all the demand of cities and agricultural areas especially those arid or semi-arid areas, due to the development of industry, agriculture and the increase of the population. The conjunctive management of surface and ground water is a subject of great practical, economic and political importance in the field of water resources. Numerous developments associated with this subject have been obtained during last two decades. An alternative management strategy is to use aquifers, the natural underground reservoirs which contain ten or hundred times more water than is held in storage in a river or in surface reservoirs. These underground reservoirs are naturally to filtering water and regulating water to some degree. Large amounts of water from precipitation or irrigation percolate down into water table as an input to aquifer. The earliest study on the interaction of river and aquifer was developed by **[11]**. He derived an analytical solution for estimation of the flow from a stream to an aquifer caused by pumping near the stream. Serrano (1998) presented a numerical model for transient stream/aquifer interactions in an alluvial valley aquifer **[8]**. The model is based on the one-dimensional Boussinesq equation for horizontal unconfined aquifer which was solved using a decomposition method. Parlange et al. (2001) developed a numerical model based on the one-dimensional Boussinesq equation for horizontal unconfined aquifer and obtain its solution by solving using finite element program. Verhoest et al. (2002) presented a numerical model and compared the results against a transient analytical solution. Recently in **[5]** Hernandez and Uddameri have developed a semi-analytical solution for stream-aquifer interactions under triangular stream-stage variations. In the present paper it is assumed that the aquifer is in contact with a drainage canal at one end of the horizontal aquifer and it is bounded by a zero flux at the impervious surface at the other end of aquifer. For one dimensional homogeneous, isotropic aquifer, without accretion the governing Boussinesq equation under Dupuit assumptions is a nonlinear partial differential equation. For specific initial and boundary conditions the nonlinear Boussinesq equation is solved using Homotopy perturbation transform method. The analysis of various parameters is studied to observe the corresponding effect on the height of water table.

2. Analysis of Homotopy perturbation transform method

We present a Homotopy perturbation transform algorithm **[1,6,10]** for solving partial differential equation written in an operator form

$$Du(x,t)+Ru(x,t)+Nu(x,t)=g(x,t) \qquad \textbf{(1)}$$

with the initial conditions

$$u(x,0)=h(x), \quad u_t(x,0)=f(x)$$

where D is the second order linear differential operator $D=\frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than D, N represents the general non-linear differential operator and $g(x,t)$ is the source term. The method consist of first applying the Laplace transform to equation (1) and then by using initial conditions, we have

$$L[Du(x,t)]+L[Ru(x,t)]+L[Nu(x,t)]=L[g(x,t)] \qquad \textbf{(2)}$$

Using Laplace transform of derivatives and applying initial conditions, we have

$$[s^2u(x,s)-su(x,0)-u_t(x,0)]+L[Ru(x,t)]+L[Nu(x,t)]=L[g(x,t)] \qquad \textbf{(3)}$$

On simplifying

$$u(x,s)=\frac{h(x)}{s}+\frac{f(x)}{s^2}-\frac{1}{s^2}L[Ru(x,t)]+\frac{1}{s^2}L[g(x,t)]-\frac{1}{s^2}L[Nu(x,t)] \qquad \textbf{(4)}$$

Applying inverse Laplace transform in equation (4), we get

$$u(x,t)=G(x,t)-L^{-1}\left[\frac{1}{s^2}L\left[Ru(x,t)+Nu(x,t)\right]\right] \qquad \textbf{(5)}$$

where $G(x,t)$ represents the term arising from source term and prescribed initial conditions. Now, we apply the homotopy perturbation method according to which u can be expanded into infinite series as

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \ , \qquad \textbf{(6)}$$

where $p \in [0,1]$ is an embedding parameter. And the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \ , \qquad \textbf{(7)}$$

where $H_n(u)$ is the He's polynomials can be generated by several means. Here we used the following recursive formulation:

$$H_n(u_0......u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}\left(p^i u_i\right)\right)\right]_{p=0} \quad ,n = 0,1,2,3,....... \qquad \textbf{(8)}$$

By substituting equation (6) and (7) in equation (5) the solution can be written as

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s^2} L\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \qquad \textbf{(9)}$$

Equating the terms with identical power of p in equation (9), we obtained the following approximations.

$$\begin{aligned} p^0 &: u_0(x,t) = G(x,t), \\ p^1 &: u_1(x,t) = -L^{-1}\left[\frac{1}{s^2} L\left[Ru_0(x,t) + H_0(u)\right]\right], \\ p^2 &: u_2(x,t) = -L^{-1}\left[-\frac{1}{s^2} L\left[Ru_1(x,t) + H_1(u)\right]\right], \\ p^3 &: u_3(x,t) = -L^{-1}\left[-\frac{1}{s^2} L\left[Ru_2(x,t) + H_2(u)\right]\right]. \end{aligned} \qquad \textbf{(10)}$$

The best approximations for the solutions are

$$u = \lim_{p\to 1} u_n = u_0 + u_1 + u_2 + ... \qquad \textbf{(11)}$$

The present method finds the solution without any discretization or restrictive assumptions and therefore reduces the numerical computation to a great extent.

3.The nonlinear Boussinesq equation

The idealized cross section of the model under consideration is depicted in figure1 below.

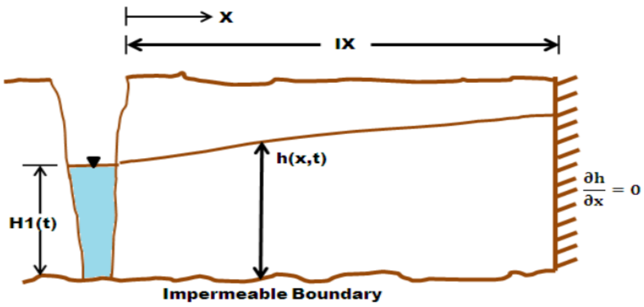

Figure1. Idealized cross section of the model of transient stream-aquifer interaction.

The governing equation for one-dimensional, lateral, unconfined groundwater flow with Dupuit assumptions is the Boussinesq equation (Bear, 1972) **[2]:**

$$\frac{\partial h}{\partial t}=\frac{K}{S_y}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) \quad , \qquad \textbf{(12)}$$

where $h(x,t)$ is the water table elevation at a distance x from the origin and time t, K, S_y are the saturated hydraulic conductivity and specific yield respectively which are considered to be constant.

In order to solve Boussinesq equation (12) completely, the specific initial condition and boundary conditions are as considered in **[8]** are given by:

$$h(0,t)=H(t), \qquad t\geq 0, \qquad \textbf{(13)}$$

$$\frac{\partial h(l_x,t)}{\partial x}=0 \ , \qquad t\geq 0, \qquad \textbf{(14)}$$

The exact solution of equation(12) – (14) is as given in **[7]**

$$h(x,t)=\frac{x}{t+1}-\frac{x^2}{6(t+1)}+\frac{3}{2(t+1)}\left[(t+1)^{2/3}-1\right] \qquad \textbf{(15)}$$

we have considered the case where $H(t)$ increases from zero to a maximum and then decreases back to zero, thus providing a realistic behaviour for all times.

We take $H(t)=\frac{3}{2(t+1)}\left[(t+1)^{2/3}-1\right]$ shown in figure 2 below.

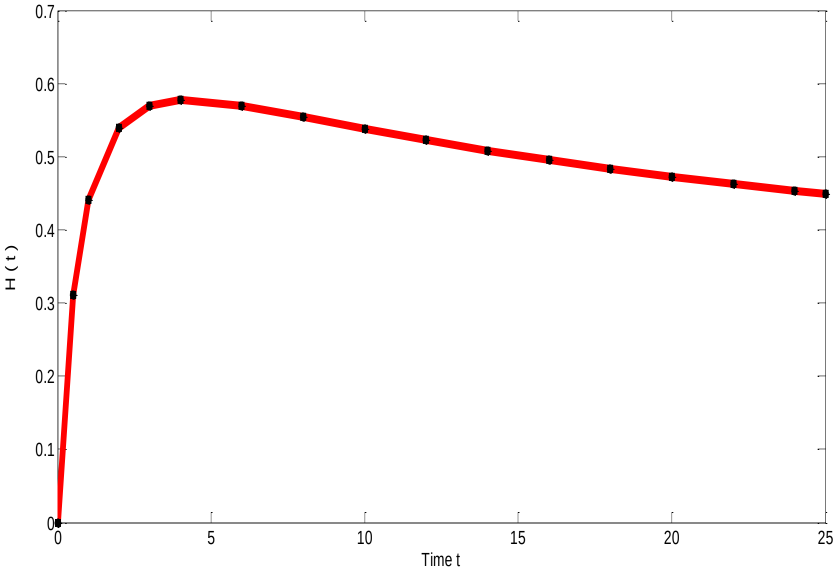

Figure 2. Schematic representation of $H(t)$.

$H(t)$ reaches a maximum equal to $\frac{1}{\sqrt{3}}$ at $t=3^{\frac{3}{2}-1}$. The interest of (13) is that the exact solution given by (15) is available for comparison. In the boundary condition(14) zero flux is considered at the impervious surface $x = l_x$.

The initial condition is expressed by

$$h(x,0)=H_0(x) \tag{16}$$

We assume initial water table in the aquifer as a quadratic approximation and choose $H_0(x)=ax^2+bx+c$.Under the above initial and boundary conditions, the solution of Boussinesq equation (12) is obtained by using Homotopy perturbation transform method.

4.Approximate analytical solution of Boussinesq equation for horizontal aquifer

To solve equation (3.28) by applying Homotopy perturbation transform method the first step is to apply Laplace transform on equation (12).

$$L\left[\frac{\partial h}{\partial t}\right]=\frac{K}{S_y}L\left[h\left(\frac{\partial^2 h}{\partial x^2}\right)+\left(\frac{\partial h}{\partial x}\right)^2\right] \tag{17}$$

This can be written as

$$[sh(x,s)-h(x,0)]=\frac{K}{S_y}L\left[h\left(\frac{\partial^2 h}{\partial x^2}\right)+\left(\frac{\partial h}{\partial x}\right)^2\right] \tag{18}$$

On applying the initial condition (16), we get

$$[sh(x,s)] = \left(a x^2 + b x + c\right) + \frac{K}{S_y} L\left[h\left(\frac{\partial^2 h}{\partial x^2}\right) + \left(\frac{\partial h}{\partial x}\right)^2 \right] \quad \textbf{(19)}$$

Taking Inverse Laplace transform, we get,

$$L^{-1}\left[h(x,s)\right] = L^{-1}\left[\frac{1}{s}\left(a x^2 + b x + c\right)\right] + \frac{K}{S_y} L^{-1}\left[\frac{1}{s} L\left[h\left(\frac{\partial^2 h}{\partial x^2}\right) + \left(\frac{\partial h}{\partial x}\right)^2 \right]\right] \quad \textbf{(20)}$$

$$h(x,t) = \left(a x^2 + b x + c\right) + \frac{K}{S_y} L^{-1}\left[\frac{1}{s} L\left[h\left(\frac{\partial^2 h}{\partial x^2}\right) + \left(\frac{\partial h}{\partial x}\right)^2 \right]\right] \quad \textbf{(21)}$$

Now we apply the Homotopy perturbation method to handle the non-linearity on the right hand side of above equation, in the form

$$h(x,t) = \sum_{n=0}^{\infty} p^n \left(h_n(x,t)\right) \quad \textbf{(22)}$$

Using Binomial expansion and He's Approximation, equation (3.37) reduces to

$$\sum_{n=0}^{\infty} p^n \left(h_n(x,t)\right) = \left(a x^2 + b x + c\right) + p\frac{K}{S_y} L^{-1}\left[\frac{1}{s} L\left[\begin{array}{l} \left(\sum_{n=0}^{\infty} p^n \left(h_n(x,t)\right)\right)\left(\left(\sum_{n=0}^{\infty} p^n \left(h_n(x,t)\right)\right)_{xx}\right) \\ + \left(\left(\sum_{n=0}^{\infty} p^n \left(h_n(x,t)\right)\right)_x\right)^2 \end{array}\right]\right] \quad \textbf{(23)}$$

This can be written in expanded form as

$$h_0 + p h_1 + p^2 h_2 + ... = \left(a x^2 + b x + c\right) + p\frac{K}{S_y} L^{-1}\left[\frac{1}{s} L\left[\begin{array}{l} (h_0 + p h_1 + p^2 h_2 + ..)\left(\frac{\partial^2 h_0}{\partial x^2} + p\frac{\partial^2 h_1}{\partial x^2} + p^2\frac{\partial^2 h_2}{\partial x^2} + ...\right) \\ + \left(\frac{\partial^2 h_0}{\partial x^2} + p\frac{\partial^2 h_1}{\partial x^2} + p^2\frac{\partial^2 h_2}{\partial x^2} + ...\right)^2 \end{array}\right]\right] \quad \textbf{(24)}$$

On comparing the coefficient of various power of p, we get

$$p^0 : h_0(x,t) = a x^2 + b x + c$$

$$p^1 : h_1(x,t) = \frac{K}{S_y} L^{-1}\left[\frac{1}{s} L\left[\left(h_0\left(\frac{\partial^2 h_0}{\partial x^2}\right) + \left(\frac{\partial h_0}{\partial x}\right)^2 \right)\right]\right]$$

$$p^2 : h_2(x,t) = \frac{K}{S_y} L^{-1}\left[\frac{1}{s} L\left[\left(h_0\left(\frac{\partial^2 h_1}{\partial x^2}\right) + h_1\left(\frac{\partial^2 h_0}{\partial x^2}\right) + \left(2\frac{\partial h_0}{\partial x}\frac{\partial h_1}{\partial x}\right)\right)\right]\right] \quad \textbf{(25)}$$

$$p^3 : h_3(x,t) = \frac{K}{S_y} L^{-1}\left[\frac{1}{s} L\left[\left(h_0 \left(\frac{\partial^2 h_2}{\partial x^2}\right) + h_1\left(\frac{\partial^2 h_1}{\partial x^2}\right) + h_2 \left(\frac{\partial^2 h_0}{\partial x^2}\right) + \left(2\frac{\partial h_0}{\partial x}\frac{\partial h_2}{\partial x} + \left(\frac{\partial h_1}{\partial x}\right)^2 \right)\right)\right]\right]$$

$$p^4 : h_4(x,t) = \frac{K}{S_y} L^{-1}\left[\frac{1}{s} L\left[\left(h_0 \left(\frac{\partial^2 h_3}{\partial x^2}\right) + h_1\left(\frac{\partial^2 h_2}{\partial x^2}\right) + h_2\left(\frac{\partial^2 h_1}{\partial x^2}\right) + h_3 \left(\frac{\partial^2 h_0}{\partial x^2}\right) + \left(2\frac{\partial h_0}{\partial x}\frac{\partial h_3}{\partial x} + 2\frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial x} \right)\right)\right]\right]$$

Proceeding in similar manner we can obtain further approximations. On solving above equations and substituting in equation (3.38) we get HPTM solution of equation (3.28) in the form of a series.

$$\begin{aligned} h(x,t) &= ax^2 + bx + c \\ &+ \frac{K}{S_y}\left(\left((b+2ax)^2 + 2a\left(ax^2+bx+c\right)\right)t\right) \\ &+ \frac{K^2}{S_y^{\ 2}}\left(\left(7b^2 + 8ac + 36abx + 36a^2x^2\right)at^2\right) \\ &+ \frac{K^3}{3S_y^{\ 3}}\left(\left(67b^2 + 56ac + 324abx + 324a^2x^2\right)2a^2t^3\right) \\ &+ \frac{K^4}{3S_y^{\ 4}}\left(52b^2 + 35ac + 243abx + 243a^2x^2\right)16a^3t^4 \end{aligned} \quad \textbf{(26)}$$

Considering the initial depth at $x = 0$ to be zero, we choose $c = 0$. Applying boundary conditions (3.29) and (3.30), we get $a = -1/6$ and $b = 1$. On substituting the value of a and b and assuming ratio $K/S_y = 1$ in equation (26) we get

$$\begin{aligned} h(x,t) &= x - \frac{x^2}{6} + t\left(\left(1-\frac{x}{3}\right)^2 + \frac{1}{3}\left(-x+\frac{x^2}{6}\right)\right) - \frac{t^2}{6}\left(7-6x+x^2\right) + \frac{t^3}{54}\left(67-54x+9x^2\right) \\ &- \frac{2t^4}{81}\left(52 - \frac{81}{2}x + \frac{27}{4}x^2\right) \end{aligned} \quad \textbf{(27)}$$

On expansion it can be written as

$$h(x,t) = t - \frac{7t^2}{6} + \frac{67t^3}{54} - \frac{104t^4}{81} + x - xt + xt^2 - xt^3 + xt^4 - \frac{x^2}{6} + \frac{x^2t}{6} - \frac{x^2t^2}{6} + \frac{x^2t^3}{6} - \frac{x^2t^4}{6} \quad \textbf{(28)}$$

On adding and subtracting a term $\left(+\frac{9}{6} - \frac{9t}{6} + \frac{9t^2}{6} - \frac{9t^3}{6} + \frac{9t^4}{6}\right)$ in equation (28), we get

$$\begin{aligned} h(x,t) &= \left(-\frac{x^2}{6} + \frac{x^2t}{6} - \frac{x^2t^2}{6} + \frac{x^2t^3}{6} - \frac{x^2t^4}{6} + x - xt + xt^2 - xt^3 + xt^4 - \frac{9}{6} + \frac{9t}{6} - \frac{9t^2}{6} + \frac{9t^3}{6} - \frac{9t^4}{6}\right) \\ &+ \left(t - \frac{7t^2}{6} + \frac{67t^3}{54} - \frac{104t^4}{81} + \frac{9}{6} - \frac{9t}{6} + \frac{9t^2}{6} - \frac{9t^3}{6} + \frac{9t^4}{6}\right) \end{aligned} \quad \textbf{(29)}$$

which can be written after rearranging the terms as,

$$h(x,t)=\frac{-(x-3)^2}{6(1+t)}+\frac{3}{2(1+t)^{1/3}} \qquad \textbf{(30)}$$

Equation (30) is the required solution of (12) obtained by using Homotopy perturbation transform method which satisfies boundary conditions (13) and (14).

5.Results and Discussion

The numerical values obtained by HPTM for specific values of time and various space coordinates are compared with the exact solution for l_x =3(m). From the table 1, it can be seen that the solution obtained by HPTM is very close to the exact solution for t = 0.25day. Its graphical representation of comparison with exact solution is given in figure 3. The numerical values at various values of time(days) and space(m) are shown in table 2. It has been observed that the numerical values obtained by HPTM converge with the exact solution. From the table 2, it is observed that the height of the water table at x = 0 increases with the time to its maximum value $\frac{1}{\sqrt{3}}$ upto t = 4.2(days) and then decreases to zero after a very long time (approximately 4000days).Also, at x = 3(m), the height of water table decreases from its initial value 1.5(m) to 0.8658(m) at t = 4.2(days) and becomes zero after a very long time (approximately 4000days). Its graphical representation is shown in figure 4 and figure 5. The decrease of the height of the water table at x = 0 is not shown in figure 4. Thus, the result satisfies the boundary conditions and behaves well with the physical phenomena for various values of time.

Table 1. Numerical values of height of water table in an unconfined horizontal aquifer obtained by Exact solution **[4]** and Homotopy perturbation transform method.

Distance (x) (m)	$h(x,t)$ at **(t =0.25days)**	
	EXACT	**HPTM**
0	0.19247665008383383	0.19247665008383374
0.2	0.34714331675050053	0.3471433167505005
0.4	0.4911433167505006	0.49114331675050027
0.6	0.6244766500838338	0.6244766500838337
0.8	0.7471433167505006	0.7471433167505003
1	0.8591433167505006	0.8591433167505004
1.2	0.9604766500838338	0.9604766500838338
1.4	1.0511433167505002	1.0511433167505002
1.6	1.1311433167505007	1.1311433167505003
1.8	1.200476650083834	1.2004766500838338
2	1.2591433167505004	1.2591433167505004
2.2	1.3071433167505004	1.3071433167505004
2.4	1.3444766500838339	1.3444766500838337
2.6	1.3711433167505005	1.3711433167505003
2.8	1.3871433167505005	1.3871433167505003
3	1.3924766500838344	1.3924766500838337

Table 2. Numerical values of water head in an unconfined horizontal aquifer for different distance x at time t = 0,…,4.3 days.

						height $h(x,t)$						
x (m)	***t*=0 (days)**	***t*=0.4 (days)**	***t*=0.8 (days)**	***t*=1.2 (days)**	***t*=1.6 (days)**	***t*=1.8 (days)**	***t*=2 (days)**	***t*=2.8 (days)**	***t*=3.2 (days)**	***t*=4 (days)**	***t*=4.2 (days)**	***t*=4.3 (days)**
0	0	0.2694	0.3998	0.4715	0.4715	0.5285	0.5400	0.5665	0.5725	0.5772	0.5774	0.5773
0.5	0.458	0.5968	0.6544	0.6798	0.6623	0.6922	0.6928	0.6871	0.6817	0.6689	0.6655	0.6638
1	0.833	0.8646	0.8627	0.8503	0.8153	0.8261	0.8178	0.7858	0.7710	0.7439	0.7376	0.7345
1.5	1.125	1.073	1.0247	0.9829	0.9304	0.9303	0.9150	0.8625	0.8404	0.8022	0.7937	0.7896
2	1.333	1.2218	1.1405	1.0776	1.0076	1.0047	0.9844	0.9174	0.8900	0.8439	0.8338	0.8289
2.5	1.458	1.3111	1.2100	1.1344	1.0469	1.0494	1.0261	0.9503	0.9198	0.8689	0.8578	0.8525
3	1.5	1.3409	1.2331	1.1533	1.0483	1.0642	1.0400	0.9612	0.9297	0.8772	0.8658	0.8603

Table 3. Numerical values of water head in an unconfined horizontal aquifer for different time *t* at distance $x = 0$, 0.5 , 1,…, 3.

				height $h(x,t)$			
t	***x*=0**	***x*=0.5**	***x*=1**	***x*=1.5**	***x*=2**	***x*=2.5**	***x*=3**
0	0	0.4583	0.8333	1.125	1.3333	1.4583	1.5
0.4	0.2694	0.5968	0.8647	1.073	1.2218	1.3111	1.3409
0.8	0.3998	0.6544	0.8627	1.0248	1.1405	1.2100	1.2331
1.2	0.4715	0.6798	0.8503	0.9829	1.0776	1.1344	1.1533
1.6	0.5139	0.6902	0.8344	0.9466	1.0267	1.0748	1.0908
2	0.5400	0.6928	0.8178	0.9150	0.9845	1.0261	1.0400
2.4	0.5564	0.6912	0.8015	0.8872	0.9485	0.9853	0.9975
2.8	0.5665	0.6871	0.7858	0.8625	0.9174	0.9503	0.9612
3.2	0.5725	0.6817	0.7710	0.8404	0.8900	0.9198	0.9297
3.6	0.5758	0.6755	0.7570	0.8204	0.8657	0.8929	0.9019
4	0.5772	0.6689	0.7439	0.8022	0.8439	0.8689	0.8772
4.2	0.5773	0.6655	0.7376	0.7937	0.8338	0.8578	0.8658

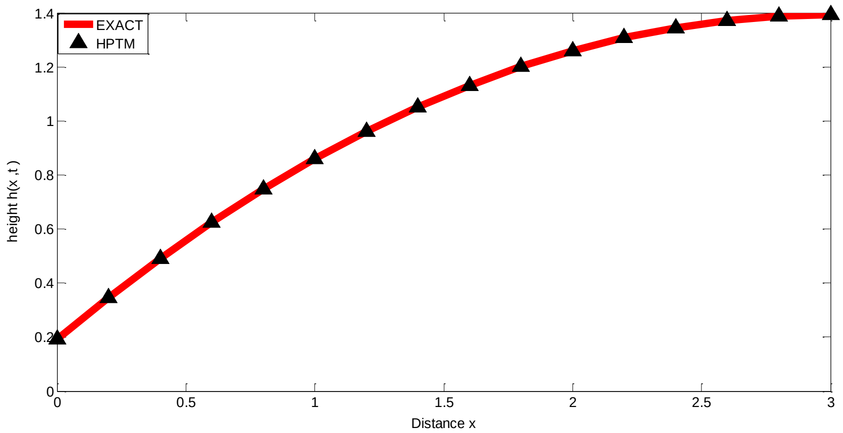

Figure 3.Graph of Exact and HPTM solution for height of water table vs distance x.

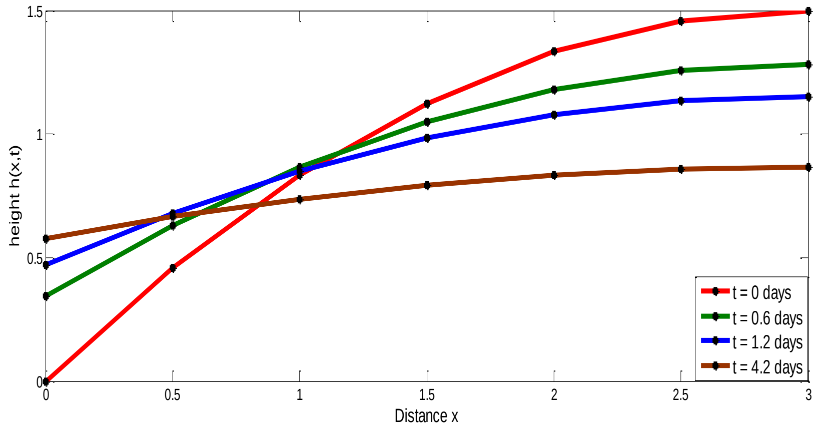

Figure 4. Height of water table in an unconfined horizontal aquifer for different distance *x* at time *t* = 0,…,4.3 days.

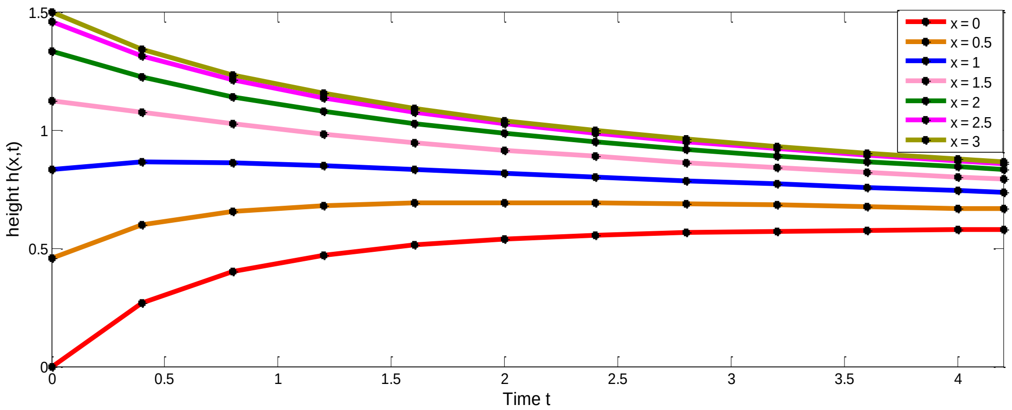

Figure 5. Height of water table in an unconfined horizontal aquifer for different time *t* at distance *x* = 0 , 0.5 , 1,…, 3.

6.Sensitivity analysis

The effect of various parameters on the height of water table in an unconfined horizontal aquifer has been observed and their numerical values and graphical representation are presented below. The numerical values of $h(x,t)$ at fixed time *t* = 0.4(day) are observed for different distance *x* by increasing the value of specific yield(S_y) keeping hydraulic conductivity(K) same. The numerical values are shown in table 4. It can be seen that there is an effect on height of water with the increase in S_y. From table 4 it is observed that with the least value of S_y(0.25) the height of the water at *x* =3(m) is decreased from its initial value 1.5m to 1.025 where as with the highest value of S_y(0.95), the height of the water table at *x*=3(m) is decreased from its initial value to 1.3402(m). A similar observation is made by keeping S_y fixed and increasing the various values of K. The numerical values under this effect are shown in table 5. It is observed that with the least value of K(1darcy) the height of the water table at *x*=3(m) is decreased from its initial value 1.5(m) to 1.0625(m)whereas with the highest value of K(1000darcy) the height of the water table at *x*=3(m) is decreased from its initial value to 0.3002(m). Thus, both the parameters are sensitive with a small change in the values of S_y and higher change in *K*. The graphical representation of the effect of S_y and K are shown in figure 6 and figure 7 respectively. It is seen that with the increasing value of S_y the height of the water is increases and with K the height of the water table decreases which is consistent with respect to the properties of the aquifer parameters. The effect of the ratio K/S_y is also observed and the numerical values of the height of water table in five different materials are calculated at specific time(*t*=0.4day). The numerical values at time *t*=0.4(day) are shown in table 6 and its graphical representation is shown is figure 8. From the graphical representation it is seen that the height of the water table is maximum between *x*=0 to *x*=3(m) in sandstone in comparison to other materials. From all the numerical values shown in table 4 to table 6 it can be seen that the changes in the height of the water table with the space variable is small. This is due to the fact that the groundwater flow is slow and aquifer does behave as a reservoir to hold the water a relatively long time.**[9]**

Table 4. Numerical values of water head in an unconfined horizontal aquifer for different value of S_y Keeping $K = 1m/day$ fixed at time *t*=0.4day.

height $h(x,t)$ at $K = 1m/day$ **(*t*=0.4 day)**				
x	$S_y = 0.25$	$S_y = 0.5$	$S_y = 0.75$	$S_y = 0.95$
0	0.2694	0.2694	0.2694	0.2694
0.5	0.4405	0.5071	0.5567	0.5893
1	0.5968	0.7150	0.7993	0.8527
1.5	0.7382	0.8931	0.9971	1.0594
2	0.8647	1.0415	1.1501	1.2096
2.5	0.9763	1.1601	1.2584	1.3032
3	1.073	1.2489	1.3219	1.3402

Table 5. Numerical values of water head in an unconfined horizontal aquifer for different value of K Keeping $S_y = 0.21$ fixed at time *t*=0.4 day.

height $h(x,t)$ $S_y = 0.21$ **(*t* =0.4 day)**				
x	$K = 1$ **m/day**	$K = 50$ **m/day**	$K = 100$ **m/day**	$K = 1000$ **m/day**
0	0.2694	0.2694	0.2694	0.2694
0.5	0.4269	0.2924	0.2857	0.2746
1	0.5718	0.3152	0.3019	0.2797
1.5	0.7042	0.3377	0.3180	0.2849
2	0.8242	0.3600	0.3339	0.2900
2.5	0.9316	0.3820	0.3497	0.2951
3	1.0265	0.4038	0.3654	0.3002

Table 6. Numerical values of water head in an unconfined horizontal aquifer for different value of ratio K/S_y **[3]** at time *t*=0.4 day.

x	$h(x,t)$ (*t*=0.4 day)				
	$K=1$ $S_y=0.02$ Sandstone	$K=1$ $S_y=0.01$ Fine Sand	$K=100$ $S_y=0.44$ Sand & gravel	$K=100$ $S_y=0.16$ Coarse Sand	$K=1000$ $S_y=0.13$ Gravel
0	0.2694	0.2694	0.2694	0.2694	0.2694
0.5	0.3193	0.3048	0.2930	0.2837	0.2735
1	0.3681	0.3397	0.3163	0.2978	0.2775
1.5	0.4156	0.3739	0.3393	0.3118	0.2816
2	0.4619	0.4075	0.3621	0.3258	0.2856
2.5	0.5071	0.4405	0.3846	0.3397	0.2897
3	0.5510	0.4730	0.4068	0.3534	0.2937

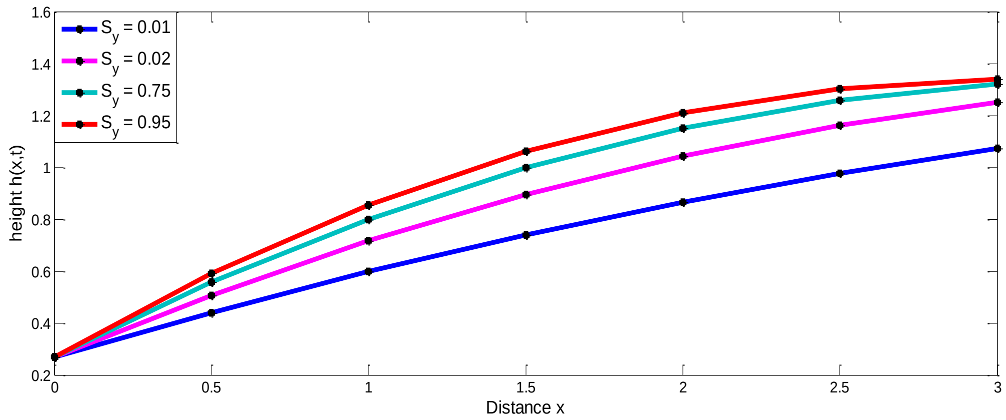

Figure 6. Height of water table in an unconfined horizontal aquifer for different value of S_y Keeping $K=1m/day$ fixed at time *t*=0.4 day.

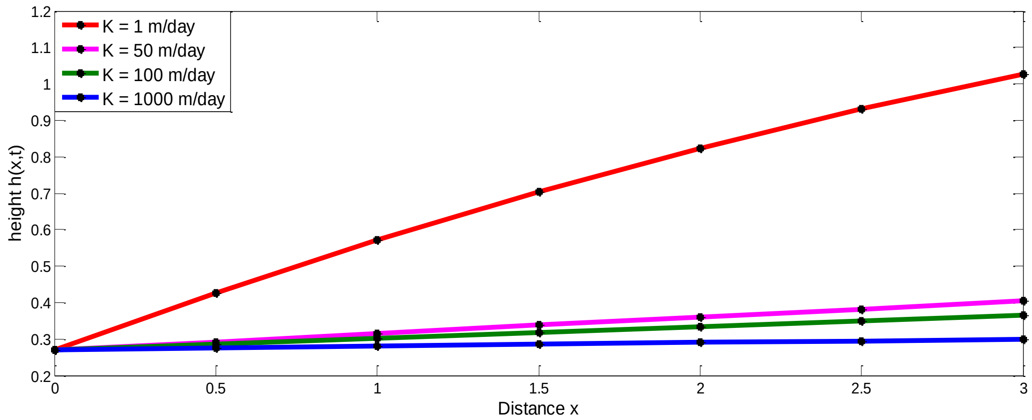

Figure 7. Height of water table in an unconfined horizontal aquifer for different value of K Keeping $S_y=0.21$ fixed at time *t*=0.4 day.

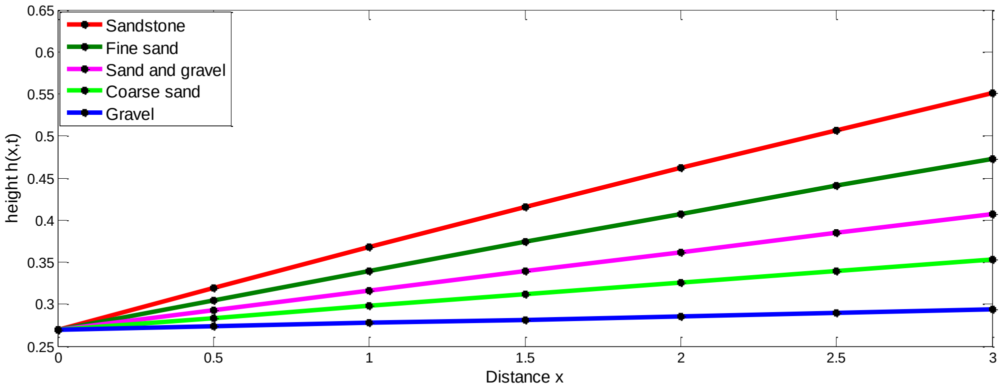

Figure 8. Height of water table in an unconfined horizontal aquifer for different values of value of ratio K/S_y at time *t*=0.4 day.

7.Conclusion

The approximate analytical solution of Boussinesq equation for horizontal aquifer is obtained by applying HPTM and compared with the available exact solution for the horizontal aquifer. We see that HPTM is easy, accurate and convenient. The combination of Homotopy perturbation method and Laplace transform overcomes the restriction of Laplace transform method to solve non-linear partial differential equation. The two important parameters viz. Hydraulic conductivity and Specific yield(S_y) are considered in the present groundwater flow problem. The approximate analytical solution is obtained by considering the ratio $\frac{K}{S_y}=1$. However, the sensitivity of these parameters is studied for five different ratios of five different samples. It is concluded that among the five samples considered the height of the water is maximum in sandstone. Various authors have obtained the solution of Boussinesq with different boundary conditions. We have shown that HPTM can be applied equally well with the suitable choice of initial condition.

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