Pseudo Bipolar Fuzzy Cosets of Bipolar Fuzzy and Bipolar Anti-Fuzzy HX Subgroups

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ABSTRACT

In this paper, we introduce the concept of pseudo bipolar fuzzy cosets, pseudo bipolar fuzzy double cosets of a bipolar fuzzy and bipolar anti-fuzzy subgroups. We also establish these concepts to bipolar fuzzy and bipolar anti-fuzzy HX subgroups of a HX group with suitable examples. Also we discuss some of their relative properties.

KEYWORDS

bipolar fuzzy set, bipolar fuzzy subgroup, bipolar anti-fuzzy subgroup, pseudo bipolar fuzzy cosets, pseudo bipolar fuzzy double cosets , bipolar fuzzy HX subgroup , bipolar anti-fuzzy HX subgroup.

1. INTRODUCTION

The notion of fuzzy sets was introduced by L.A. Zadeh [15]. Fuzzy set theory has been developed in many directions by many researchers and has evoked great interest among mathematicians working in different fields of mathematics. In 1971, Rosenfeld [12] introduced the concept of fuzzy subgroup. R. Biswas [1] introduced the concept of anti fuzzy subgroup of group. Li Hongxing [2] introduced the concept of HX group and the authors Chengzhong et al.[4] introduced the concept of fuzzy HX group. Palaniappan.N. et al.[12] discussed the concepts of anti-fuzzy group and its Lower level subgroups. Muthuraj.R.,et al.[7],[9] discussed the concepts of bipolar fuzzy subgroups and bipolar anti fuzzy subgroups and also discussed bipolar fuzzy HX subgroup and its level sub HX groups, bipolar anti-fuzzy HX subgroups and its lower level sub HX groups. Bhattacharya [10] introduced fuzzy right coset and fuzzy left coset of a group. B. Vasantha kandasamy [14] introduced the concept of pseudo fuzzy cosets, and pseudo fuzzy double cosets of a fuzzy group of a group. In this paper we define the concept of pseudo bipolar fuzzy cosets, pseudo bipolar fuzzy double cosets of bipolar fuzzy and bipolar anti-fuzzy subgroups of a group. Also introduce the concept of pseudo bipolar fuzzy cosets and pseudo bipolar fuzzy double cosets of bipolar fuzzy and bipolar anti-fuzzy HX subgroups of a HX group and study some of their related properties.

2. PRELIMINARIES

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, G = (G, *) is a group, e is the identity element of G, and xy, we mean x * y.

2.1 Definition [15]

Let X be any non empty set. A fuzzy subset μ of X is a function $\mu: X \to [0, 1]$.

2.2 Definition [1]

A fuzzy set μ on G is called fuzzy subgroup of G if for $x, y \in G$,

i. $\mu(xy) \geq \min \{\mu(x), \mu(y)\}$

ii. $\mu(x^{-1}) = \mu(x)$.

2.3 Definition [1]

A fuzzy set μ on G is called an anti- fuzzy subgroup of G if for $x, y \in G$,

i. $\mu(xy) \leq \max \{\mu(x), \mu(y)\}$

ii. $\mu(x^{-1}) = \mu(x)$.

2.4 Definition [9]

Let G be a non-empty set. A bipolar-valued fuzzy set or bipolar fuzzy set μ in G is an object having the form $\mu=\{\langle x, \mu^{+}(x), \mu^{-}(x)\rangle : x \in G\}$ where $\mu^{+} : G \to [0,1]$ and $\mu^{-} : G \to [-1,0]$ are mappings. The positive membership degree $\mu^{+}(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\mu =\{\langle x, \mu^{+}(x), \mu^{-}(x)\rangle: x \in G\}$ and the negative membership degree $\mu^{-}(x)$ denotes the satisfaction degree of an element x to some implicit counter property corresponding to a bipolar-valued fuzzy set $\mu = \{\langle x, \mu^{+}(x), \mu^{-}(x) \rangle : x \in G\}$. If $\mu^{+}(x) \neq 0$ and $\mu^{-}(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\mu = \{\langle x, \mu^{+}(x), \mu^{-}(x)\rangle : x \in G\}$. If $\mu^{+}(x) = 0$ and $\mu^{-}(x) \neq 0$, it is the situation that x does not satisfy the property of $\mu =\{\langle x, \mu^{+}(x), \mu^{-}(x)\rangle: x \in G\}$, but somewhat satisfies the counter property of $\mu = \{\langle x, \mu^{+}(x), \mu^{-}(x)\rangle : x \in G\}$. It is possible for an element x to be such that $\mu^{+}(x) \neq 0$ and $\mu^{-}(x) \neq 0$ when the membership function of property overlaps that its counter property over some portion of G. For the sake of simplicity, we shall use the symbol $\mu = (\mu^{+},\mu^{-})$ for the bipolar-valued fuzzy set $\mu = \{\langle x, \mu^{+}(x), \mu^{-}(x)\rangle: x \in G\}$.

2.5 Definition [9]

A bipolar-valued fuzzy set or bipolar fuzzy set $\mu = (\mu^{+}, \mu^{-})$ is called a bipolar fuzzy subgroup of G if for $x, y \in G$,

i. $\mu^{+}(xy) \geq \min \{\mu^{+}(x), \mu^{+}(y)\}$

ii. $\mu^{-}(xy) \leq \max \{\mu^{-}(x), \mu^{-}(y)\}$

iii. $\mu^{+}(x^{-1}) = \mu^{+}(x)$, $\mu^{-}(x^{-1}) = \mu^{-}(x)$.

2.6 Definition [9]

A bipolar-valued fuzzy set or bipolar fuzzy set $\mu = (\mu^{+}, \mu^{-})$ is called a bipolar anti-fuzzy subgroup of G if for $x, y \in G$,

i. $\mu^{+}(xy) \leq \max \{\mu^{+}(x), \mu^{+}(y)\}$

ii. $\mu^{-}(xy) \geq \min \{\mu^{-}(x), \mu^{-}(y)\}$

iii. $\mu^{+}(x^{-1}) = \mu^{+}(x)$, $\mu^{-}(x^{-1}) = \mu^{-}(x)$.

3. Pseudo Bipolar Fuzzy Cosets And Pseudo Bipolar Fuzzy Double Cosets Of Bipolar Fuzzy And Bipolar Anti-Fuzzy Subgroups And Their Properties:

In this section, we define the concepts of pseudo bipolar fuzzy cosets, pseudo bipolar fuzzy double cosets of a bipolar fuzzy and bipolar anti-fuzzy subgroups of a group and discuss some of their properties.

3.1 Definition

Let $\mu = (\mu^+,\mu^-)$ be a bipolar fuzzy subgroup of a group G and let $a \in G$. Then the pseudo bipolar fuzzy coset $(a\mu)^P = (((a\mu)^P)^+ , ((a\mu)^P)^-)$ is defined by

i. $((a\mu)^P)^+ (x) = |p(a)|\, \mu^+ (x)$
ii. $((a\mu)^P)^- (x) = |p(a)|\, \mu^- (x)$,for every $x \in G$

and for some $p \in P$, where $P = \{p(x) / p(x) \in [-1,1]$ and $p (x) \neq 0$ for all $x \in G \}$.

3.2 Example

Let G be a Klein's four group. Then $G = \{e,a,b,ab\}$ where $a^2 = e = b^2$, $ab = ba$ and e is the identity element of G. Define a bipolar fuzzy subset $\mu = (\mu^+,\mu^-)$ on G as,

$$\mu^+(x) = \begin{cases} 0.6 \text{ ,if } x = e \\ 0.4 \text{ ,if } x = a \\ 0.3 \text{ ,if } x = b, ab \end{cases} \qquad \mu^-(x) = \begin{cases} -0.7 \text{ , if } x = e \\ -0.6 \text{ ,if } x = a \\ -0.4 \text{ ,if } x = b, ab \end{cases}$$

Let us take p as follows

$$p(x) = \begin{cases} 0.8 \text{ ,if } x = e \\ 0.6 \text{ ,if } x = a \\ 0.4 \text{ ,if } x = b \\ 0.3 \text{ ,if } x = ab \end{cases}$$

Now we calculate the pseudo bipolar fuzzy coset of $\mu = (\mu^+,\mu^-)$ .For the identity element e of the group G, we have $(e\mu)^p = \mu$.

For the element a of G, we have

$$((a\mu)^p)^+(x) = |p(a)|\ \mu^+(x) = \begin{cases} 0.36 \text{ ,if } x = e \\ 0.24 \text{ ,if } x = a \\ 0.18 \text{ ,if } x = b, ab \end{cases}$$

$$((a\mu)^{p})^{-}(x) = |p(a)|\ \mu^{-}(x) = \begin{cases} -0.42 \ ,\text{if} \ \ x = e \\ -0.36 \ ,\text{if} \ \ x = a \\ -0.24 \ ,\text{if} \ \ x = b, ab \end{cases}$$

For the element b of G, we have

$$((b\mu)^{p})^{+}(x) = |p(b)|\mu^{+}(x) = \begin{cases} 0.24 \ ,\text{if} \ \ x = e \\ 0.16 \ ,\text{if} \ \ x = a \\ 0.12 \ ,\text{if} \ \ x = b, ab \end{cases}$$

$$((b\mu)^{p})^{-}(x) = |p(b)|\ \mu^{-}(x) = \begin{cases} -0.28 \ ,\text{if} \ \ x = e \\ -0.24 \ ,\text{if} \ \ x = a \\ -0.16 \ ,\text{if} \ \ x = b, ab \end{cases}$$

For the element ab of G, we have

$$((ab\mu)^{p})^{+}(x) = |p(ab)|\ \mu^{+}(x) = \begin{cases} 0.18 \ ,\text{if} \ \ x = e \\ 0.12 \ ,\text{if} \ \ x = a \\ 0.09 \ ,\text{if} \ \ x = b, ab \end{cases}$$

$$((ab\mu)^{p})^{-}(x) = |p(ab)|\ \mu^{-}(x) = \begin{cases} -0.21 \ ,\text{if} \ \ x = e \\ -0.18 \ ,\text{if} \ \ x = a \\ -0.12 \ ,\text{if} \ \ x = b, ab \end{cases}$$

Note: The pseudo bipolar fuzzy cosets of $\mu = (\mu^{+}, \mu^{-})$ are bipolar fuzzy subgroups of G since $\mu^{+}(e) \geq \mu^{+}(x)$ and $\mu^{-}(e) \leq \mu^{-}(x)$.

3.3 Definition

Let $\mu = (\mu^{+}, \mu^{-})$ be a bipolar anti-fuzzy subgroup of a group G and let $a \in G$. Then the pseudo bipolar fuzzy coset $(a\mu)^{P} = (((a\mu)^{P})^{+}, ((a\mu)^{P})^{-})$ is defined by

i. $((a\mu)^{P})^{+}(x) = |p(a)|\ \mu^{+}(x)$

ii. $((a\mu)^{P})^{-}(x) = |p(a)|\ \mu^{-}(x)$, for every $x \in G$ and for some $p \in P$, Where $P = \{p(x) / p(x) \in [-1,1]$ and $p(x) \neq 0$ for all $x \in G\}$.

3.4 Example

Let G be a Klein's four group. Then G = {e,a,b,ab} where $a^2 = e = b^2$, $ab = ba$ and e is the identity element of G. Define a bipolar fuzzy subset $\mu = (\mu^{+}, \mu^{-})$ on G as,

$$\mu^+(x)=\begin{cases}0.3\ ,\text{if } \ x=e\\ 0.5\ ,\text{if } \ x=a\\ 0.7\ ,\text{if } \ x=b,ab\end{cases}\qquad \mu^-(x)=\begin{cases}-0.4\ ,\text{if } \ x=e\\ -0.6\ ,\text{if } \ x=a\\ -0.8\ ,\text{if } \ x=b,ab\end{cases}$$

Clearly $\mu=(\mu^+,\mu^-)$ is a bipolar anti-fuzzy subgroup of G.

Let us take p as follows

$$p(x)=\begin{cases}-0.6\ ,\text{if } \ x=e\\ -0.5\ ,\text{if } \ x=a\\ -0.7\ ,\text{if } \ x=b\\ -0.8\ ,\text{if } \ x=ab\end{cases}$$

Now we calculate the pseudo bipolar fuzzy coset of $\mu=(\mu^+,\mu^-)$.For the identity element e of the group G ,we have $(e\mu)^p=\mu$.

For the element a of G, we have

$$((a\mu)^p)^+(x)=|p(a)|\ \mu^+(x)=\begin{cases}0.15\ ,\text{if } \ x=e\\ 0.25\ ,\text{if } \ x=a\\ 0.35\ ,\text{if } \ x=b,ab\end{cases}$$

$$((a\mu)^p)^-(x)=|p(a)|\ \mu^-(x)=\begin{cases}-0.20\ ,\text{if } \ x=e\\ -0.30\ ,\text{if } \ x=a\\ -0.40\ ,\text{if } \ x=b,ab\end{cases}$$

For the element b of G, we have

$$((b\mu)^p)^+(x)=|p(b)|\ \mu^+(x)=\begin{cases}0.21\ ,\text{if } \ x=e\\ 0.35\ ,\text{if } \ x=a\\ 0.49\ ,\text{if } \ x=b,ab\end{cases}$$

$$((b\mu)^p)^-(x)=|p(b)|\ \mu^-(x)=\begin{cases}-0.28\ ,\text{if } \ x=e\\ -0.42\ ,\text{if } \ x=a\\ -0.56\ ,\text{if } \ x=b,ab\end{cases}$$

For the element ab of G, we have

$$((ab\mu)^p)^+(x) = |p(ab)|\,\mu^+(x) = \begin{cases} 0.24 \ ,\text{if} \ \ x = e \\ 0.4 \ ,\text{if} \ \ x = a \\ 0.56 \ ,\text{if} \ \ x = b, ab \end{cases}$$

$$((ab\mu)^p)^-(x) = |p(ab)|\,\mu^-(x) = \begin{cases} -0.32 \ ,\text{if} \ \ x = e \\ -0.48 \ ,\text{if} \ \ x = a \\ -0.64 \ ,\text{if} \ \ x = b, ab \end{cases}$$

Note: The pseudo bipolar fuzzy cosets of $\mu = (\mu^+, \mu^-)$ are bipolar anti-fuzzy subgroups of G since $\mu^+(e) \le \mu^+(x)$ and $\mu^-(e) \ge \mu^-(x)$.

3.5 Definition

Let μ and φ be any two bipolar fuzzy subgroups of a group G, then pseudo bipolar fuzzy double coset $(\mu a\varphi)^p$ is defined by

i. $((\mu a\varphi)^p)^+ = ((a\mu)^p \cap (a\varphi)^p)^+$
ii. $((\mu a\varphi)^p)^- = ((a\mu)^p \cap (a\varphi)^p)^-$ where $a \in G$ for some $p \in P$.

3.6 Example

Let $G = \{1, -1, i, -i\}$ be a group under the binary operation multiplication. Let bipolar fuzzy subsets $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ on G are defined as,

$$\mu^+(x) = \begin{cases} 0.7 \ ,\text{if} \ \ x = 1 \\ 0.6 \ ,\text{if} \ \ x = -1 \\ 0.4 \ ,\text{if} \ \ x = i, -i \end{cases} \qquad \mu^-(x) = \begin{cases} -0.6 \ ,\text{if} \ \ x = 1 \\ -0.4 \ ,\text{if} \ \ x = -1 \\ -0.3 \ ,\text{if} \ \ x = i, -i \end{cases}$$

$$\varphi^+(x) = \begin{cases} 0.8 \ ,\text{if} \ \ x = 1 \\ 0.5 \ ,\text{if} \ \ x = -1 \\ 0.3 \ ,\text{if} \ \ x = i, -i \end{cases} \qquad \varphi^-(x) = \begin{cases} -0.7 \ ,\text{if} \ \ x = 1 \\ -0.3 \ ,\text{if} \ \ x = -1 \\ -0.2 \ ,\text{if} \ \ x = i, -i \end{cases}$$

Clearly $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ are bipolar fuzzy subgroups of G.

Let us take p as follows $p(x) = -0.3$ for every $x \in G$, then the pseudo bipolar fuzzy cosets are,

$$((a\mu)^p)^+(x) = |p(a)|\,\mu^+(x) = \begin{cases} 0.21 \ ,\text{if} \ \ x = 1 \\ 0.18 \ ,\text{if} \ \ x = -1 \\ 0.12 \ ,\text{if} \ \ x = i, -i \end{cases}$$

$$((a\mu)^p)^-(x) = |p(a)|\ \mu^-(x) = \begin{cases} -0.18\ ,\text{if}\ \ x=1 \\ -0.12\ ,\text{if}\ \ x=-1 \\ -0.09\ ,\text{if}\ \ x=i,-i \end{cases}$$

$$((a\varphi)^p)^+(x) = |p(a)|\ \varphi^+(x) = \begin{cases} 0.24\ ,\text{if}\ \ x=1 \\ 0.15\ ,\text{if}\ \ x=-1 \\ 0.09\ ,\text{if}\ \ x=i,-i \end{cases}$$

$$((a\varphi)^p)^-(x) = |p(a)|\ \varphi^-(x) = \begin{cases} -0.21\ ,\text{if}\ \ x=1 \\ -0.09\ ,\text{if}\ \ x=-1 \\ -0.06\ ,\text{if}\ \ x=i,-i \end{cases}$$

Now the pseudo bipolar fuzzy double cosets are

$$((\mu a\varphi)^p)^+(x) = \begin{cases} 0.21\ ,\text{if}\ \ x=1 \\ 0.15\ ,\text{if}\ \ x=-1 \\ 0.09\ ,\text{if}\ \ x=i,-i \end{cases} \qquad ((\mu a\varphi)^p)^-(x) = \begin{cases} -0.18\ ,\text{if}\ \ x=1 \\ -0.09\ ,\text{if}\ \ x=-1 \\ -0.06\ ,\text{if}\ \ x=i,-i \end{cases}$$

3.7 Definition

Let μ and φ be any two bipolar anti-fuzzy subgroups of a group G, then pseudo bipolar fuzzy double coset $(\mu a\varphi)^p$ is defined by

i. $((\mu a\varphi)^p)^+ = ((a\mu)^p \cap (a\varphi)^p)^+$
ii. $((\mu a\varphi)^p)^- = ((a\mu)^p \cap (a\varphi)^p)^-$ where $a\in G$ for some $p\in P$.

3.8 Example

Let $G=\{1,-1,i,-i\}$ be a group under the binary operation multiplication. Define bipolar fuzzy subsets $\mu=(\mu^+,\mu^-)$ and $\varphi=(\varphi^+,\varphi^-)$ on G as,

$$\mu^+(x) = \begin{cases} 0.4\ ,\text{if}\ \ x=1 \\ 0.6\ ,\text{if}\ \ x=-1 \\ 0.7\ ,\text{if}\ \ x=i,-i \end{cases} \qquad \mu^-(x) = \begin{cases} -0.3\ ,\text{if}\ \ x=1 \\ -0.4\ ,\text{if}\ \ x=-1 \\ -0.6\ ,\text{if}\ \ x=i,-i \end{cases}$$

$$\varphi^+(x) = \begin{cases} 0.3\ ,\text{if}\ \ x=1 \\ 0.5\ ,\text{if}\ \ x=-1 \\ 0.8\ ,\text{if}\ \ x=i,-i \end{cases} \qquad \varphi^-(x) = \begin{cases} -0.2\ ,\text{if}\ \ x=1 \\ -0.5\ ,\text{if}\ \ x=-1 \\ -0.7\ ,\text{if}\ \ x=i,-i \end{cases}$$

Clearly, $\mu=(\mu^+,\mu^-)$ and $\varphi=(\varphi^+,\varphi^-)$ are bipolar anti-fuzzy subgroups of G.

Let us take p as follows p(x) = 0.2 for every $x \in G$, then the pseudo bipolar fuzzy cosets are,

$$((a\mu)^p)^+(x) = |p(a)|\mu^+(x) = \begin{cases} 0.08 \text{ ,if } x = 1 \\ 0.12 \text{ ,if } x = -1 \\ 0.14 \text{ ,if } x = i, -i \end{cases}$$

$$((a\mu)^p)^-(x) = |p(a)|\ \mu^-(x) = \begin{cases} -0.06 \text{ ,if } x = 1 \\ -0.08 \text{ ,if } x = -1 \\ -0.12 \text{ ,if } x = i, -i \end{cases}$$

$$((a\varphi)^p)^+(x) = |p(a)|\ \varphi^+(x) = \begin{cases} 0.06 \text{ ,if } x = 1 \\ 0.10 \text{ ,if } x = -1 \\ 0.16 \text{ ,if } x = i, -i \end{cases}$$

$$((a\varphi)^p)^-(x) = |p(a)|\ \varphi^-(x) = \begin{cases} -0.04 \text{ ,if } x = 1 \\ -0.1 \text{ ,if } x = -1 \\ -0.14 \text{ ,if } x = i, -i \end{cases}$$

Now the pseudo bipolar fuzzy double cosets are

$$((\mu a\varphi)^p)^+(x) = \begin{cases} 0.08 \text{ ,if } x = 1 \\ 0.12 \text{ ,if } x = -1 \\ 0.16 \text{ ,if } x = i, -i \end{cases} \qquad ((\mu a\varphi)^p)^-(x) = \begin{cases} -0.06 \text{ ,if } x = 1 \\ -0.1 \text{ ,if } x = -1 \\ -0.14 \text{ ,if } x = i, -i \end{cases}$$

3.9 Theorem

If $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subgroup of a group G, then the pseudo bipolar fuzzy coset $(a\mu)^P = (((a\mu)^P)^+, ((a\mu)^P)^-)$ is a bipolar fuzzy subgroup of a group G.

Proof: Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subgroup of a group G.

For every x and y in G, we have,

i. $((a\mu)^P)^+(xy^{-1}) = |p(a)|\ \mu^+(xy^{-1})$
$\geq |p(a)| \min\{\mu^+(x), \mu^+(y)\}$
$= \min\{|p(a)|\ \mu^+(x), |p(a)|\ \mu^+(y)\}$
$= \min\{((a\mu)^P)^+(x), ((a\mu)^P)^+(y)\}$
Therefore, $((a\mu)^P)^+(xy^{-1}) \geq \min\{((a\mu)^P)^+(x), ((a\mu)^P)^+(y)\}$

ii. $((a\mu)^P)^-(xy^{-1}) = |p(a)|\ \mu^-(xy^{-1})$
$\leq |p(a)| \max\{\mu^-(x), \mu^-(y)\}$

$= \max\{|p(a)|\,\mu^-(x), |p(a)|\,\mu^-(y)\}$
$= \max\{((a\mu)^P)^-(x), ((a\mu)^P)^-(y)\}$
Therefore, $((a\mu)^P)^-(xy^{-1}) \le \max\{((a\mu)^P)^-(x), ((a\mu)^P)^-(y)\}$
Hence $(a\mu)^P$ is a bipolar fuzzy subgroup of a group G.

Note: $(a\mu)^P$ is called as a pseudo bipolar fuzzy subgroup of a group G if $|p(a)| \le |p(e)|$, for every $a \in G$.

3.10 Theorem

If $\mu = (\mu^+, \mu^-)$ be a bipolar anti-fuzzy subgroup of a group G, then the pseudo bipolar fuzzy coset $(a\mu)^P = (((a\mu)^P)^+, ((a\mu)^P)^-)$ is a bipolar anti-fuzzy subgroup of a group G.

Proof: It is clear from the Theorem 3.9

Note: $(a\mu)^P$ is called as a pseudo bipolar anti-fuzzy subgroup of a group G if $|p(a)| \ge |p(e)|$, for every $a \in G$.

3.11 Theorem

If $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ are two bipolar fuzzy subgroups of a group G, then the pseudo bipolar fuzzy double coset $(\mu a\varphi)^P = (((\mu a\varphi)^P)^+, ((\mu a\varphi)^P)^-)$ determined by μ and φ is also a bipolar fuzzy subgroup of the group G.

Proof: For all $x,y \in G$,

i. $((\mu a\varphi)^P)^+(xy^{-1}) = \{((a\mu)^P \cap (a\varphi)^P)^+\}(xy^{-1})$
$= \min\{((a\mu)^P)^+(xy^{-1}), ((a\varphi)^P)^+(xy^{-1})\}$
$= \min\{|p(a)|\,\mu^+(xy^{-1}), |p(a)|\,\varphi^+(xy^{-1})\}$
$= |p(a)| \min\{\mu^+(xy^{-1}), \varphi^+(xy^{-1})\}$
$\ge |p(a)| \min\{\min\{\mu^+(x), \mu^+(y)\}, \min\{\varphi^+(x), \varphi^+(y)\}\}$
$= |p(a)| \min\{\min\{\mu^+(x), \varphi^+(x)\}, \min\{\mu^+(y), \varphi^+(y)\}\}$
$= \min\{\min\{|p(a)|\,\mu^+(x), |p(a)|\,\varphi^+(x)\}, \min\{|p(a)|\,\mu^+(y), |p(a)|\,\varphi^+(y)\}\}$
$= \min\{\min\{((a\mu)^P)^+(x),((a\varphi)^P)^+(x)\},\min\{((a\mu)^P)^+(y),((a\varphi)^P)^+(y)\}\}$
$= \min\{((a\mu)^P \cap (a\varphi)^P)^+(x), ((a\mu)^P \cap (a\varphi)^P)^+(y)\}$
$= \min\{((\mu a\varphi)^P)^+(x), ((\mu a\varphi)^P)^+(y)\}$

ii. $((\mu a\varphi)^P)^-(xy^{-1}) = \{((a\mu)^P \cap (a\varphi)^P)^-\}(xy^{-1})$
$= \max\{((a\mu)^P)^-(xy^{-1}), ((a\varphi)^P)^-(xy^{-1})\}$
$= \max\{|p(a)|\,\mu^-(xy^{-1}), |p(a)|\,\varphi^-(xy^{-1})\}$
$= |p(a)| \max\{\mu^-(xy^{-1}), \varphi^-(xy^{-1})\}$
$\le |p(a)| \max\{\max\{\mu^-(x), \mu^-(y)\}, \max\{\varphi^-(x), \varphi^-(y)\}\}$
$= |p(a)| \max\{\max\{\mu^-(x), \varphi^-(x)\}, \max\{\mu^-(y), \varphi^-(y)\}\}$
$= \max\{\max\{|p(a)|\,\mu^-(x), |p(a)|\,\varphi^-(x)\}, \max\{|p(a)|\,\mu^-(y), |p(a)|\,\varphi^-(y)\}\}$

$= \max\{\max\{((a\mu)^P)^-(x), ((a\varphi)^P)^-(x)\}, \max\{((a\mu)^P)^-(y), ((a\varphi)^P)^-(y)\}\}$
$= \max\{((a\mu)^P \cap (a\varphi)^P)^-(x), ((a\mu)^P \cap (a\varphi)^P)^-(y)\}$
$= \max\{((\mu a\varphi)^P)^-(x), ((\mu a\varphi)^P)^-(y)\}$

Hence, $(\mu a\varphi)^P = (((\mu a\varphi)^P)^+, ((\mu a\varphi)^P)^-)$ is a bipolar fuzzy subgroup of the group G.

3.12 Theorem

If $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ are two bipolar anti-fuzzy subgroups of a group G, then the pseudo bipolar fuzzy double coset $(\mu a\varphi)^P = (((\mu a\varphi)^P)^+, ((\mu a\varphi)^P)^-)$ is also a bipolar anti- fuzzy subgroup of the group G.

Proof: For all $x,y \in G$,

i. $((\mu a\varphi)^P)^+(xy^{-1}) = \{((a\mu)^P \cap (a\varphi)^P)^+\}(xy^{-1})$
$= \max\{((a\mu)^P)^+(xy^{-1}), ((a\varphi)^P)^+(xy^{-1})\}$
$= \max\{|p(a)|\,\mu^+(xy^{-1}), |p(a)|\,\varphi^+(xy^{-1})\}$
$= |p(a)| \max\{\mu^+(xy^{-1}), \varphi^+(xy^{-1})\}$
$\leq |p(a)| \max\{\max\{\mu^+(x), \mu^+(y)\}, \max\{\varphi^+(x), \varphi^+(y)\}\}$
$= |p(a)| \max\{\max\{\mu^+(x), \varphi^+(x)\}, \max\{\mu^+(y), \varphi^+(y)\}\}$
$= \max\{\max\{|p(a)|\,\mu^+(x), |p(a)|\,\varphi^+(x)\}, \max\{|p(a)|\,\mu^+(y), |p(a)|\,\varphi^+(y)\}\}$
$= \max\{\max\{((a\mu)^P)^+(x), ((a\varphi)^P)^+(x)\}, \max\{((a\mu)^P)^+(y), ((a\varphi)^P)^+(y)\}\}$
$= \max\{((a\mu)^P \cap (a\varphi)^P)^+(x), ((a\mu)^P \cap (a\varphi)^P)^+(y)\}$
$= \max\{((\mu a\varphi)^P)^+(x), ((\mu a\varphi)^P)^+(y)\}$

ii. $((\mu a\varphi)^P)^-(xy^{-1}) = \{((a\mu)^P \cap (a\varphi)^P)^-\}(xy^{-1})$
$= \min\{((a\mu)^P)^-(xy^{-1}), ((a\varphi)^P)^-(xy^{-1})\}$
$= \min\{|p(a)|\,\mu^-(xy^{-1}), |p(a)|\,\varphi^-(xy^{-1})\}$
$= |p(a)| \min\{\mu^-(xy^{-1}), \varphi^-(xy^{-1})\}$
$\geq |p(a)| \min\{\min\{\mu^-(x), \mu^-(y)\}, \min\{\varphi^-(x), \varphi^-(y)\}\}$
$= |p(a)| \min\{\min\{\mu^-(x), \varphi^-(x)\}, \min\{\mu^-(y), \varphi^-(y)\}\}$
$= \min\{\min\{|p(a)|\,\mu^-(x), |p(a)|\,\varphi^-(x)\}, \min\{|p(a)|\,\mu^-(y), |p(a)|\,\varphi^-(y)\}\}$
$= \min\{\min\{((a\mu)^P)^-(x), ((a\varphi)^P)^-(x)\}, \min\{((a\mu)^P)^-(y), ((a\varphi)^P)^-(y)\}\}$
$= \min\{((a\mu)^P \cap (a\varphi)^P)^-(x), ((a\mu)^P \cap (a\varphi)^P)^-(y)\}$
$= \min\{((\mu a\varphi)^P)^-(x), ((\mu a\varphi)^P)^-(y)\}$

Hence, $(\mu a\varphi)^P = (((\mu a\varphi)^P)^+, ((\mu a\varphi)^P)^-)$ is a bipolar anti-fuzzy subgroup of the group G.

4. Pseudo bipolar fuzzy cosets and Pseudo bipolar fuzzy double cosets of bipolar fuzzy and bipolar anti-fuzzy HX subgroups and their properties:

In this section, we define the concepts of pseudo bipolar fuzzy cosets, pseudo bipolar fuzzy double cosets of a bipolar fuzzy and bipolar anti-fuzzy HX subgroups of a HX group and discuss some of their properties.

4.1 Definition [2]

Let G be a finite group. In $2^G - \{\phi\}$, a nonempty set $\vartheta \subset 2^G - \{\phi\}$ is called a HX group on G, if ϑ is a group with respect to the algebraic operation defined by AB = { ab /a $\in$ A and b $\in$ B}, which its unit element is denoted by E.

4.2 Definition [9]

Let ϑ be a non-empty set. A bipolar-valued fuzzy set or bipolar fuzzy set λ_μ in ϑ is an object having the form $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$ where $\lambda_\mu^+ : \vartheta \to [0,1]$ and $\lambda_\mu^- : \vartheta \to [-1,0]$ are mappings. The positive membership degree $\lambda_\mu^+(A)$ denotes the satisfaction degree of an element A to the property corresponding to a bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle: A \in \vartheta \}$ and the negative membership degree $\lambda_\mu^-(A)$ denotes the satisfaction degree of an element A to some implicit counter property corresponding to a bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle: A \in \vartheta\}$. If $\lambda_\mu^+(A) \neq 0$ and $\lambda_\mu^-(A) = 0$, it is the situation that A is regarded as having only positive satisfaction for $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle: A \in \vartheta\}$. If $\lambda_\mu^+(A) = 0$ and $\lambda_\mu^-(A) \neq 0$, it is the situation that A does not satisfy the property of $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$, but somewhat satisfies the counter property of $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle: A\in\vartheta\}$. It is possible for an element A to be such that $\lambda_\mu^+(A) \neq 0$ and $\lambda_\mu^-(A) \neq 0$ when the membership function of property overlaps that its counter property over some portion of ϑ. For the sake of simplicity, we shall use the symbol $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ for the bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle: A \in \vartheta\}$.

4.3 Definition [9]

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group of G. A bipolar fuzzy set $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ defined on ϑ is said to be a bipolar fuzzy subgroup induced by μ on ϑ or a bipolar fuzzy HX subgroup of ϑ if for A,B $\in\vartheta$,

i. $\lambda_\mu^+(AB) \geq \min\{ \lambda_\mu^+(A), \lambda_\mu^+(B)\}$
ii. $\lambda_\mu^-(AB) \leq \max\{ \lambda_\mu^-(A), \lambda_\mu^-(B)\}$
iii. $\lambda_\mu^+(A^{-1}) = \lambda_\mu^+(A)$, $\lambda_\mu^-(A^{-1}) = \lambda_\mu^-(A)$.

Where $\lambda_\mu^+(A) = \max \{\mu^+(x) / \text{for all } x \in A \subseteq G\}$ and
$\lambda_\mu^-(A) = \min \{\mu^-(x) / \text{for all } x \in A \subseteq G\}$.

Remark [9]

i. If μ is a bipolar fuzzy subgroup of G then the bipolar fuzzy subset $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ is a fuzzy HX subgroup on ϑ.
ii. Let μ be a bipolar fuzzy subset of a group G. If $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ is a bipolar fuzzy HX subgroup on ϑ, then μ need not be a bipolar fuzzy subgroup of G.

4.4 Definition [9]

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group of G. A bipolar fuzzy set $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ defined on ϑ is said to be a bipolar anti- fuzzy subgroup induced by μ on ϑ or a bipolar anti-fuzzy HX subgroup of ϑ if for A, B$\in\vartheta$,

i. $\lambda_\mu^+(AB) \leq \max\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$
ii. $\lambda_\mu^-(AB) \geq \min\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$
iii. $\lambda_\mu^+(A^{-1}) = \lambda_\mu^+(A)$, $\lambda_\mu^-(A^{-1}) = \lambda_\mu^-(A)$.

Where $\lambda_\mu^+(A) = \min\{\mu^+(x) / \text{for all } x \in A \subseteq G\}$ and
$\lambda_\mu^-(A) = \max\{\mu^-(x) / \text{for all } x \in A \subseteq G\}$

Remark [7]

1.If $\mu = (\mu^+, \mu^-)$ is a bipolar anti-fuzzy subgroup of G then $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ is a bipolar anti-fuzzy HX subgroup on ϑ.

2.Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset of a group G. If $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ is a bipolar anti-fuzzy HX subgroup on ϑ,then $\mu = (\mu^+, \mu^-)$ need not be a bipolar anti-fuzzy subgroup of G.

4.5 Definition

Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of a HX group ϑ and $A \in \vartheta$. Then the pseudo bipolar fuzzy coset $(A\lambda_\mu)^P = (((A\lambda_\mu)^P)^+, ((A\lambda_\mu)^P)^-)$ is defined by

i. $((A\lambda_\mu)^P)^+(X) = |p(A)|\, \lambda_\mu^+(X)$
ii. $((A\lambda_\mu)^P)^-(X) = |p(A)|\, \lambda_\mu^-(X)$.For every $X \in \vartheta$ and
iii.

for some $p \in P$ Where $P = \{p(X) / p(X) \in [-1,1] \text{ and } p(X) \neq 0 \text{ for all } X \in \vartheta\}$.

4.6 Example

Let $G = \{1, -1, i, -i\}$ be a group under the binary operation multiplication. Define a bipolar fuzzy subset $\mu = (\mu^+, \mu^-)$ on G as,

$$\mu^+(x) = \begin{cases} 0.8 \text{ ,if } x = 1 \\ 0.7 \text{ ,if } x = -1 \\ 0.5 \text{ ,if } x = i, -i \end{cases} \qquad \mu^-(x) = \begin{cases} -0.7 \text{ , if } x = 1 \\ -0.6 \text{ ,if } x = -1 \\ -0.4 \text{ ,if } x = i, -i \end{cases}$$

Clearly $\mu = (\mu^+, \mu^-)$ is a bipolar fuzzy subgroup of G.

Let $\vartheta = \{\{1, -1\}, \{i, -i\}\} = \{E, A\}$, where $E = \{1, -1\}$, $A = \{i, -i\}$ .Clearly $(\vartheta, \cdot)$ is a HX group. Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy subset on ϑ induced by $\mu = (\mu^+, \mu^-)$ on G is,

$$\lambda_\mu^+(X) = \begin{cases} 0.8 \text{ ,if } X = E \\ 0.5 \text{ ,if } X = A \end{cases} \qquad \lambda_\mu^-(X) = \begin{cases} -0.7 \text{ , if } X = E \\ -0.4 \text{ ,if } X = A \end{cases}$$

Let us take p as follows: $p(X) = \begin{cases} 0.5 & ,\text{if } X = E \\ 0.66 & ,\text{if } X = A \end{cases}$

the pseudo bipolar fuzzy cosets are,

i. $$((E\lambda_\mu)^p)^+(X) = |p(E)|\ \lambda_\mu{}^+(X) = \begin{cases} 0.40 & ,\text{if } X = E \\ 0.25 & ,\text{if } X = A \end{cases}$$

$$((E\lambda_\mu)^p)^-(X) = |p(E)|\ \lambda_\mu{}^-(X) = \begin{cases} -0.35 & ,\text{if } X = E \\ -0.20 & ,\text{if } X = A \end{cases}$$

ii. $$((A\lambda_\mu)^p)^+(X) = |p(A)|\ \lambda_\mu{}^+(X) = \begin{cases} 0.528 & ,\text{if } X = E \\ 0.132 & ,\text{if } X = A \end{cases}$$

$$((A\lambda_\mu)^p)^-(X) = |p(A)|\ \lambda_\mu{}^-(X) = \begin{cases} -0.452 & ,\text{if } X = E \\ -0.264 & ,\text{if } X = A \end{cases}$$

Note: If $p(X) = \pm 1$, for all $X \in \vartheta$, then $(X\lambda_\mu)^p = \lambda_\mu$.

4.7 Definition

Let $\lambda_\mu = (\lambda_\mu{}^+, \lambda_\mu{}^-)$ be a bipolar anti-fuzzy HX subgroup of a HX group ϑ and let $A \in \vartheta$. Then the pseudo bipolar fuzzy coset $(A\lambda_\mu)^P = (((A\lambda_\mu)^P)^+, ((A\lambda_\mu)^P)^-)$ is defined by

i. $((A\lambda_\mu)^P)^+(X) = |p(A)|\ \lambda_\mu{}^+(X)$
ii. $((A\lambda_\mu)^P)^-(X) = |p(A)|\ \lambda_\mu{}^-(X)$
iii.

for every $X \in \vartheta$ and for some $p \in P$, Where $P = \{ p(X) \ / \ p(X) \in [-1,1]$ and $p(X) \neq 0$ for all $X \in \vartheta\}$.

4.8 Example

Let G = { e,a,b,ab } be a group under the binary operation multiplication. Where $a^2 = e = b^2$, ab = ba and e is the identity element of G .Define a bipolar fuzzy subset $\mu = (\mu^+, \mu^-)$ on G as,

$$\mu^+(x) = \begin{cases} 0.4 & ,\text{if } x = e \\ 0.5 & ,\text{if } x = a \\ 0.8 & ,\text{if } x = b, ab \end{cases} \qquad \mu^-(x) = \begin{cases} -0.5 & ,\text{if } x = e \\ -0.6 & ,\text{if } x = a \\ -0.7 & ,\text{if } x = b, ab \end{cases}$$

Clearly $\mu = (\mu^+, \mu^-)$ is a bipolar anti fuzzy subgroup of G.

Let $\vartheta = \{\{e,a\},\{b,ab\}\} = \{ E,A \}$, where E = {e,a} , A = {b,ab} .Clearly $(\vartheta, \cdot)$ is a HX group.

Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy subset on ϑ induced by $\mu = (\mu^+, \mu^-)$ on G is,

$$\lambda_\mu^+(X) = \begin{cases} 0.4 \ , \text{if} \ \ X = E \\ 0.8 \ , \text{if} \ \ X = A \end{cases} \qquad \lambda_\mu^-(X) = \begin{cases} -0.5 \ , \text{if} \ \ X = E \\ -0.7 \ , \text{if} \ \ X = A \end{cases}$$

Let us take p as follows: $p(X) = \begin{cases} 0.6 \quad , \text{if} \ \ X = E \\ 0.5 \quad , \text{if} \ \ X = A \end{cases}$

the pseudo bipolar fuzzy cosets are,

i. $$((E\lambda_\mu)^p)^+(X) = |p(E)| \, \lambda_\mu^+(X) = \begin{cases} 0.24 \ , \text{if} \ \ X = E \\ 0.48 \ , \text{if} \ \ X = A \end{cases}$$

$$((E\lambda_\mu)^p)^-(X) = |p(E)| \, \lambda_\mu^-(X) = \begin{cases} -0.3 \quad , \text{if} \ \ X = E \\ -0.42 \ , \text{if} \ \ X = A \end{cases}$$

ii. $$((A\lambda_\mu)^p)^+(X) = |p(A)| \, \lambda_\mu^+(X) = \begin{cases} 0.2 \ , \text{if} \ \ X = E \\ 0.4 \ , \text{if} \ \ X = A \end{cases}$$

$$((A\lambda_\mu)^p)^-(X) = |p(A)| \, \lambda_\mu^-(X) = \begin{cases} -0.25 \ , \text{if} \ \ X = E \\ -0.35 \ , \text{if} \ \ X = A \end{cases}$$

Note: If $p(X) = \pm 1$ and $p(X) \neq 0$, for all $X \in \vartheta$, then $(X\,\lambda_\mu)^p = \lambda_\mu$.

4.9 Definition

Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ be any two bipolar fuzzy HX subgroups of a HX group ϑ, then pseudo bipolar fuzzy double coset $(\lambda_\mu A \sigma_\varphi)^P = (((\lambda_\mu A\sigma_\varphi)^P)^+, ((\lambda_\mu A\sigma_\varphi)^P)^-)$ is defined by

i. $((\lambda_\mu A\sigma_\varphi)^P)^+(X) = ((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+(X) = \min\{((A\lambda_\mu)^P)^+(X) , ((A\sigma_\varphi)^P)^+(X)\}$

ii. $((\lambda_\mu A\sigma_\varphi)^P)^-(X) = ((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^-(X) = \max\{((A\lambda_\mu)^P)^-(X), ((A\sigma_\varphi)^P)^-(X)\}$

for every $X \in \vartheta$ and for some $p \in P$, Where $P = \{p(X) \,/\, p(X) \in [-1,1] , p(X) \neq 0$ for all $X \in \vartheta\}$.

4.10 Example

Let $G = \{1, -1, i, -i\}$ be a group under the binary operation multiplication. Define a bipolar fuzzy subsets $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ on G as,

$$\mu^+(x) = \begin{cases} 0.8 \ , \text{if} \ \ x = 1 \\ 0.6 \ , \text{if} \ \ x = -1 \\ 0.3 \ , \text{if} \ \ x = i, -i \end{cases} \qquad \mu^-(x) = \begin{cases} -0.6 \ , \text{if} \ \ x = 1 \\ -0.4 \ , \text{if} \ \ x = -1 \\ -0.2 \ , \text{if} \ \ x = i, -i \end{cases}$$

$$\varphi^{+}(x)=\begin{cases}0.9 \text{ ,if } x=1\\ 0.5 \text{ ,if } x=-1\\ 0.2 \text{ ,if } x=i,-i\end{cases} \qquad \varphi^{-}(x)=\begin{cases}-0.8 \text{ , if } x=1\\ -0.6 \text{ ,if } x=-1\\ -0.1 \text{ ,if } x=i,-i\end{cases}$$

and let $\vartheta = \{\{1, -1\},\{i, -i\}\} = \{E,A\}$, where $E = \{1, -1\}$, $A = \{i, -i\}$.Clearly $(\vartheta, \cdot)$ is a HX group. Let $\lambda_\mu = (\lambda_\mu^{+}, \lambda_\mu^{-})$ and $\sigma_\varphi = (\sigma_\varphi^{+}, \sigma_\varphi^{-})$ are bipolar fuzzy subsets on ϑ induced by $\mu=(\mu^{+},\mu^{-})$ and $\varphi=(\varphi^{+},\varphi^{-})$ on G are

$$\lambda_\mu^{+}(X)=\begin{cases}0.8 \text{ ,if } X=E\\ 0.3 \text{ ,if } X=A\end{cases} \qquad \lambda_\mu^{-}(X)=\begin{cases}-0.6 \text{ , if } X=E\\ -0.2 \text{ ,if } X=A\end{cases}$$

$$\sigma_\varphi^{+}(X)=\begin{cases}0.9 \text{ ,if } X=E\\ 0.2 \text{ ,if } X=A\end{cases} \qquad \sigma_\varphi^{-}(X)=\begin{cases}-0.8 \text{ , if } X=E\\ -0.1 \text{ ,if } X=A\end{cases}$$

Then the pseudo fuzzy cosets $(A\lambda_\mu)^p$ and $(A\sigma_\varphi)^p$ for $p(X) = -0.4$, for every $X \in \vartheta$ is

i. $$((A\lambda_\mu)^p)^{+}(X) = |p(A)|\,\lambda_\mu^{+}(X) = \begin{cases}0.32 \text{ ,if } X=E\\ 0.12 \text{ ,if } X=A\end{cases}$$

$$((A\lambda_\mu)^p)^{-}(X) = |p(A)|\,\lambda_\mu^{-}(X) = \begin{cases}-0.24 \text{ ,if } X=E\\ -0.08 \text{ ,if } X=A\end{cases}$$

ii. $$((A\sigma_\varphi)^p)^{+}(X) = |p(A)|\,\sigma_\varphi^{+}(X) = \begin{cases}0.36 \text{ ,if } X=E\\ 0.08 \text{ ,if } X=A\end{cases}$$

$$((A\sigma_\varphi)^p)^{-}(X) = |p(A)|\,\sigma_\varphi^{-}(X) = \begin{cases}-0.32 \text{ ,if } X=E\\ -0.04 \text{ ,if } X=A\end{cases}$$

and Now the pseudo bipolar fuzzy double coset determined by λ_μ and σ_φ is given by

$$\left(\left(\lambda_\mu A\sigma_\varphi\right)^p\right)^{+}(X) = \begin{cases}0.32 \text{ ,if } X=E\\ 0.08 \text{ ,if } X=A\end{cases} , \quad \left(\left(\lambda_\mu A\sigma_\varphi\right)^p\right)^{-}(X) = \begin{cases}-0.24 \text{ ,if } X=E\\ -0.04 \text{ ,if } X=A\end{cases}$$

Note:

Similarly we can define pseudo bipolar fuzzy double cosets on a bipolar anti-fuzzy HX subgroup of a HX group.

4.11 Theorem

If $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of a HX group ϑ, then the pseudo bipolar fuzzy coset $(A\lambda_\mu)^P = (((A\lambda_\mu)^P)^+, ((A\lambda_\mu)^P)^-)$ is a bipolar fuzzy HX subgroup of a HX group ϑ.

Proof: Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of a HX group ϑ.

For every X and Y in ϑ, we have,

i. $((A\lambda_\mu)^P)^+ (XY^{-1}) = |p(A)|\ \lambda_\mu^+ (XY^{-1})$
$\geq |p(A)| \min \{ \lambda_\mu^+ (X), \lambda_\mu^+ (Y)\}$
$= \min \{ |p(A)| \lambda_\mu^+ (X), |p(A)|\ \lambda_\mu^+ (Y)\}$
$= \min \{((A\lambda_\mu)^P)^+ (X), ((A\lambda_\mu)^P)^+ (Y)\}$
Therefore, $((A\lambda_\mu)^P)^+ (XY^{-1}) \geq \min \{((A\lambda_\mu)^P)^+ (X), ((A\lambda_\mu)^P)^+ (Y)\}$

ii. $((A\lambda_\mu)^P)^- (XY^{-1}) = |p(A)|\ \lambda_\mu^- (XY^{-1})$
$\leq |p(A)|\ \max\{ \lambda_\mu^- (X), \lambda_\mu^- (Y)\}$
$= \max \{|p(A)|\ \lambda_\mu^- (X), |p(A)|\ \lambda_\mu^- (Y)\}$
$= \max \{((A\lambda_\mu)^P)^- (X), ((A\lambda_\mu)^P)^- (Y)\}$
Therefore, $((A\lambda_\mu)^P)^- (XY^{-1}) \leq \max\{((A\lambda_\mu)^P)^- (X), ((A\lambda_\mu)^P)^- (Y)\}$
Hence $(A\lambda_\mu)^P$ is a bipolar fuzzy HX subgroup of a HX group ϑ.

Note: $(A\lambda_\mu)^P$ is called as a pseudo bipolar fuzzy HX subgroup of ϑ if $|p(A)| \leq |p(E)|$ for every $A \in \vartheta$.

4.12 Theorem

If $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti-fuzzy HX subgroup of a HX group ϑ, then the pseudo bipolar fuzzy coset $(A\lambda_\mu)^P$ is also a bipolar anti-fuzzy HX subgroup of a HX group ϑ.

Proof: it is clear from Theorem 4.11.

Note: $(A\lambda_\mu)^P$ is also called as pseudo bipolar anti-fuzzy HX subgroup of ϑ if $|p(A)| \geq |p(E)|$ for every $A \in \vartheta$.

4.13 Theorem

If $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ are two bipolar fuzzy HX subgroups of a HX group ϑ, then the pseudo bipolar fuzzy double coset $(\lambda_\mu A\sigma_\varphi)^P = (((\lambda_\mu A\sigma_\varphi)^P)^+, ((\lambda_\mu A\sigma_\varphi)^P)^-)$ determined by λ_μ and σ_φ is also a bipolar fuzzy HX subgroup of the HX group ϑ.

Proof: For all $X, Y \in \vartheta$,

i. $((\lambda_\mu A\sigma_\varphi)^P)^+(XY^{-1}) = \{((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+\}(XY^{-1})$

$= \min \{ ((A\lambda_\mu)^P)^+ (XY^{-1}) , ((A\sigma_\varphi)^P)^+ (XY^{-1})\}$
$= \min \{ |p(A)| \lambda_\mu^+ (XY^{-1}) , |p(A)| \sigma_\varphi^+ (XY^{-1})\}$
$= |p(A)| \min \{ \lambda_\mu^+ (XY^{-1}) , \sigma_\varphi^+ (XY^{-1})\}$
$\geq |p(A)| \min \{\min \{ \lambda_\mu^+ (X), \lambda_\mu^+ (Y)\}, \min\{ \sigma_\varphi^+ (X), \sigma_\varphi^+ (Y) \}\}$
$= |p(A)| \min \{\min \{ \lambda_\mu^+ (X) , \sigma_\varphi^+ (X)\}, \min\{ \lambda_\mu^+ (Y) , \sigma_\varphi^+ (Y)\}\}$
$= \min\{\min\{ |p(A)| \lambda_\mu^+ (X) , |p(A)| \sigma_\varphi^+ (X)\}, \min \{ |p(A)| \lambda_\mu^+ (Y), |p(A)| \sigma_\varphi^+ (Y)\}\}$
$= \min\{\min\{((A\lambda_\mu)^P)^+(X),((A\sigma_\varphi)^P)^+(X)\},\min\{((A\lambda_\mu)^P)^+(X),((A\sigma_\varphi)^P)^+(X)\}\}$
$= \min \{((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+ (X) , ((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+ (Y)\}$
$= \min \{((\lambda_\mu A\sigma_\varphi)^P)^+ (X) , ((\lambda_\mu A\sigma_\varphi)^P)^+ (Y) \}$

ii. $((\lambda_\mu A\sigma_\varphi)^P)^-(XY^{-1}) = \{((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^- \}(XY^{-1})$
$= \max \{ ((A\lambda_\mu)^P)^- (XY^{-1}) , ((A\sigma_\varphi)^P)^- (XY^{-1})\}$
$= \max \{ |p(A)| \lambda_\mu^- (XY^{-1}) , |p(A)| \sigma_\varphi^- (XY^{-1})\}$
$= |p(A)| \max \{ \lambda_\mu^- (XY^{-1}) , \sigma_\varphi^- (XY^{-1})\}$
$\leq |p(A)| \max \{\max \{ \lambda_\mu^- (X), \lambda_\mu^- (Y)\}, \max\{ \sigma_\varphi^- (X), \sigma_\varphi^- (Y) \}\}$
$= |p(A)| \max \{\max \{ \lambda_\mu^- (X), \sigma_\varphi^- (X)\}, \max\{ \lambda_\mu^- (Y) , \sigma_\varphi^- (Y)\}\}$
$= \max\{\max\{|p(A)| \lambda_\mu^- (X) , |p(A)| \sigma_\varphi^- (X)\}, \max\{|p(A)| \lambda_\mu^-(Y), |p(A)| \sigma_\varphi^- (Y)\}\}$
$= \max \{\max\{((A\lambda_\mu)^P)^- (X), ((A\sigma_\varphi)^P)^- (X)\}, \max\{((A\lambda_\mu)^P)^- (Y), ((A\sigma_\varphi)^P)^- (Y)\}\}$
$= \max \{((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^- (X) , ((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^- (Y)\}$
$= \max \{((\lambda_\mu A\sigma_\varphi)^P)^- (X) , ((\lambda_\mu A\sigma_\varphi)^P)^- (Y) \}$

Hence, $(\lambda_\mu A\sigma_\varphi)^P = (((\lambda_\mu A\sigma_\varphi)^P)^+,((\lambda_\mu A\sigma_\varphi)^P)^-)$ is bipolar fuzzy HX subgroup of the HX group ϑ.

4.14 Theorem

If $\lambda_\mu = (\lambda_\mu^+ , \lambda_\mu^-)$ and $\sigma_\varphi = (\sigma_\varphi^+ , \sigma_\varphi^-)$ are two bipolar anti-fuzzy HX subgroups of a HX group ϑ, then the pseudo bipolar fuzzy double coset $(\lambda_\mu A\sigma_\varphi)^P = (((\lambda_\mu A\sigma_\varphi)^P)^+, ((\lambda_\mu A\sigma_\varphi)^P)^-)$ determined by λ_μ and σ_φ is also a bipolar anti-fuzzy HX subgroup of the HX group ϑ.

Proof : For all X, Y $\in \vartheta$,

i. $((\lambda_\mu A\sigma_\varphi)^P)^+(XY^{-1}) = \{((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+ \}(XY^{-1})$
$= \min \{ ((A\lambda_\mu)^P)^+ (XY^{-1}) , ((A\sigma_\varphi)^P)^+ (XY^{-1})\}$
$= \min \{ |p(A)| \lambda_\mu^+ (XY^{-1}) , |p(A)| \sigma_\varphi^+ (XY^{-1})\}$
$= |p(A)| \min \{ \lambda_\mu^+ (XY^{-1}) , \sigma_\varphi^+ (XY^{-1})\}$
$\leq |p(A)| \min \{\max \{ \lambda_\mu^+ (X), \lambda_\mu^+ (Y)\}, \max\{ \sigma_\varphi^+ (X), \sigma_\varphi^+ (Y) \}\}$
$= |p(A)| \max \{\min \{ \lambda_\mu^+ (X) , \sigma_\varphi^+ (X)\}, \min\{ \lambda_\mu^+ (Y) , \sigma_\varphi^+ (Y)\}\}$
$= \max\{\min\{ |p(A)| \lambda_\mu^+ (X) , |p(A)| \sigma_\varphi^+ (X)\}, \min \{ |p(A)| \lambda_\mu^+ (Y) , |p(A)| \sigma_\varphi^+ (Y)\}\}$
$= \max\{\min\{((A\lambda_\mu)^P)^+(X),((A\sigma_\varphi)^P)^+(X)\},\min\{((A\lambda_\mu)^P)^+(Y),((A\sigma_\varphi)^P)^+(Y)\}\}$
$= \max \{((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+ (X) , ((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+ (Y)\}$
$= \max \{((\lambda_\mu A\sigma_\varphi)^P)^+ (X) , ((\lambda_\mu A\sigma_\varphi)^P)^+ (Y) \}$

ii. $((\lambda_\mu A\sigma_\varphi)^P)^-(XY^{-1}) = \{((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^- \}(XY^{-1})$
$= \max \{ ((A\lambda_\mu)^P)^- (XY^{-1}) , ((A\sigma_\varphi)^P)^- (XY^{-1})\}$
$= \max \{ |p(A)| \lambda_\mu^- (XY^{-1}) , |p(A)| \sigma_\varphi^- (XY^{-1})\}$
$= |p(A)| \max \{ \lambda_\mu^- (XY^{-1}) , \sigma_\varphi^- (XY^{-1})\}$
$\geq |p(A)| \max \{\min \{ \lambda_\mu^- (X), \lambda_\mu^- (Y)\}, \min\{ \sigma_\varphi^- (X), \sigma_\varphi^- (Y) \}\}$
$= |p(A)| \min \{\max \{ \lambda_\mu^- (X), \sigma_\varphi^- (X)\}, \max\{ \lambda_\mu^- (Y) , \sigma_\varphi^- (Y)\}\}$
$= \min\{\max\{|p(A)| \lambda_\mu^- (X), |p(A)| \sigma_\varphi^- (X)\}, \max\{|p(A)| \lambda_\mu^-(Y), |p(A)| \sigma_\varphi^- (Y)\}\}$
$= \min \{\max\{((A\lambda_\mu)^P)^- (X), ((A\sigma_\varphi)^P)^- (X)\}, \max\{((A\lambda_\mu)^P)^- (Y), ((A\sigma_\varphi)^P)^- (Y)\}\}$

$= \min \{((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^-(X), ((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^-(Y)\}$
$= \min \{((\lambda_\mu A\sigma_\varphi)^P)^-(X), ((\lambda_\mu A\sigma_\varphi)^P)^-(Y)\}$

Hence, $(\lambda_\mu A\sigma_\varphi)^P = (((\lambda_\mu A\sigma_\varphi)^P)^+, ((\lambda_\mu A\sigma_\varphi)^P)^-)$ is a bipolar anti-fuzzy HX subgroup of the HX group ϑ.

4.15 Theorem

Intersection of any two pseudo bipolar fuzzy HX subgroup on a HX subgroup ϑ is also a pseudo bipolar fuzzy HX subgroup of ϑ.

Proof: Let $(A\lambda_\mu)^P = (((A\lambda_\mu)^P)^+, ((A\lambda_\mu)^P)^-)$ and $(A\sigma_\varphi)^P = (((A\sigma_\varphi)^P)^+, ((A\sigma_\varphi)^P)^-)$ be any two pseudo bipolar fuzzy HX subgroups of a HX group ϑ.Now, Intersection of any two pseudo bipolar fuzzy HX subgroup on a HX subgroup ϑ is $((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P) = (((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+, ((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^-)$

i. $((A\lambda_\mu)^P \cap (A\eta_\varphi)^P)^+(XY^{-1}) = \min \{((A\lambda_\mu)^P)^+(XY^{-1}), ((A\sigma_\varphi)^P)^+(XY^{-1})\}$

$= \min \{ |p(A)|\, \lambda_\mu^+(XY^{-1}), p^+(A)\, \sigma_\varphi^+(XY^{-1})\}$
$= |p(A)| \min\{ \lambda_\mu^+(XY^{-1}), \sigma_\varphi^+(XY^{-1})\}$
$\geq |p(A)| \min\{\min\{ \lambda_\mu^+(X), \lambda_\mu^+(Y)\}, \min\{ \sigma_\varphi^+(X), \sigma_\varphi^+(Y)\}\}$
$= |p(A)| \min\{\min\{ \lambda_\mu^+(X), \sigma_\varphi^+(X)\}, \min\{ \lambda_\mu^+(Y), \sigma_\varphi^+(Y)\}\}$
$= \min\{ |p(A)|\{ \min\{ \lambda_\mu^+(X), \sigma_\varphi^+(X)\}, \min\{ \lambda_\mu^+(Y), \sigma_\varphi^+(Y)\}\}\}$
$= \min\{\min\{ |p(A)|\, \lambda_\mu^+(X), |p(A)|\, \sigma_\varphi^+(X)\}, \min\{ |p(A)|\, \lambda_\mu^+(Y), |p(A)|\, \sigma_\varphi^+(Y)\}\}$
$= \min \{\min\{((A\lambda_\mu)^P)^+(X), ((A\sigma_\varphi)^P)^+(X)\}, \min \{((A\lambda_\mu)^P)^+(Y), ((A\sigma_\varphi)^P)^+(Y)\}\}$
$= \min \{((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+(X), ((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+(Y)\}\}$

i. $((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^-(XY^{-1}) = \max \{((A\lambda_\mu)^P)^-(XY^{-1}), ((A\sigma_\varphi)^P)^-(XY^{-1})\}$
$= \max \{ |p(A)|\, \lambda_\mu^-(XY^{-1}), |p(A)|\, \sigma_\varphi^-(XY^{-1})\}$
$= |p(A)| \max\{ \lambda_\mu^-(XY^{-1}), \sigma_\varphi^-(XY^{-1})\}$
$\leq |p(A)| \max\{\max\{ \lambda_\mu^-(X), \lambda_\mu^-(Y)\}, \max\{ \sigma_\varphi^-(X), \sigma_\varphi^-(Y)\}\}$
$= |p(A)| \max\{\max\{ \lambda_\mu^-(X), \sigma_\varphi^-(X)\}, \max\{ \lambda_\mu^-(Y), \sigma_\varphi^-(Y)\}\}$
$= \max\{|p(A)|\{\max\{ \lambda_\mu^-(X),\sigma_\varphi^-(X)\}, \max\{\lambda_\mu^-(Y),\sigma_\varphi^-(Y)\}\}\}$
$= \max\{\max\{|p(A)|\, \lambda_\mu^-(X), |p(A)|\, \sigma_\varphi^-(X)\},\max\{|p(A)|\, \lambda_\mu^-(Y), |p(A)|\, \sigma_\varphi^-(Y)\}\}$
$= \max \{\max\{((A\lambda_\mu)^P)^-(X), ((A\sigma_\varphi)^P)^-(X)\},\max\{((A\lambda_\mu)^P)^-(Y), ((A\sigma_\varphi)^P)^-(Y)\}\}$
$= \max \{((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^-(X), ((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^-(Y)\}\}$

Hence the intersection of any two pseudo bipolar fuzzy HX subgroups of a HX group ϑ is also a pseudo bipolar fuzzy HX subgroup of ϑ.

4.16 Theorem

Intersection of any two pseudo bipolar anti-fuzzy HX subgroup on a HX subgroup ϑ is also a pseudo bipolar anti-fuzzy HX subgroup of ϑ.

Proof: Let $(A\lambda_\mu)^P = (((A\lambda_\mu)^P)^+, ((A\lambda_\mu)^P)^-)$ and $(A\sigma_\varphi)^P = (((A\sigma_\varphi)^P)^+, ((A\sigma_\varphi)^P)^-)$ be any two pseudo bipolar anti-fuzzy HX subgroups of a HX group ϑ.

Now, Intersection of any two pseudo bipolar anti-fuzzy HX subgroup on a HX subgroup ϑ is $((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P) = (((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+, ((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^-)$

i. $((A\lambda_\mu)^P \cap (A\eta_\varphi)^P)^+ (XY^{-1}) = \min\{((A\lambda_\mu)^P)^+ (XY^{-1}), ((A\sigma_\varphi)^P)^+ (XY^{-1})\}$
$= \min\{ |p(A)| \lambda_\mu^+ (XY^{-1}), |p(A)| \sigma_\varphi^+ (XY^{-1})\}$
$= |p(A)| \min\{ \lambda_\mu^+ (XY^{-1}), \sigma_\varphi^+ (XY^{-1})\}$
$\leq |p(A)| \min\{\max\{ \lambda_\mu^+ (X), \lambda_\mu^+ (Y)\}, \max\{ \sigma_\varphi^+ (X), \sigma_\varphi^+ (Y)\}\}$
$= |p(A)| \max\{\min\{ \lambda_\mu^+ (X), \sigma_\varphi^+ (X)\}, \min\{ \lambda_\mu^+ (Y), \sigma_\varphi^+ (Y)\}\}$
$= \max\{ |p(A)|\{\min\{ \lambda_\mu^+ (X), \sigma_\varphi^+ (X)\}, \min\{ \lambda_\mu^+ (Y), \sigma_\varphi^+ (Y)\}\}\}$
$= \max\{\min\{ |p(A)| \lambda_\mu^+(X), |p(A)| \sigma_\varphi^+(X)\}, \min\{ |p(A)| \lambda_\mu^+(Y), |p(A)| \sigma_\varphi^+(Y)\}\}$
$= \max\{\min\{((A\lambda_\mu)^P)^+ (X), ((A\sigma_\varphi)^P)^+ (X)\}, \min\{((A\lambda_\mu)^P)^+ (Y), ((A\sigma_\varphi)^P)^+ (Y)\}\}$
$= \max\{((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+ (X), ((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^+ (Y)\}\}$

i. $((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^- (XY^{-1}) = \max\{((A\lambda_\mu)^P)^- (XY^{-1}), ((A\sigma_\varphi)^P)^- (XY^{-1})\}$
$= \max\{ |p(A)| \lambda_\mu^- (XY^{-1}), |p(A)| \sigma_\varphi^- (XY^{-1})\}$
$= |p(A)| \max\{ \lambda_\mu^- (XY^{-1}), \sigma_\varphi^- (XY^{-1})\}$
$\geq |p(A)| \max\{\min\{ \lambda_\mu^- (X), \lambda_\mu^- (Y)\}, \min\{ \sigma_\varphi^- (X), \sigma_\varphi^- (Y)\}\}$
$= |p(A)| \min\{\max\{ \lambda_\mu^- (X), \sigma_\varphi^- (X)\}, \max\{ \lambda_\mu^- (Y), \sigma_\varphi^- (Y)\}\}$
$= \min\{ |p(A)|\{ \max\{ \lambda_\mu^- (X), \sigma_\varphi^- (X)\}, \max\{ \lambda_\mu^- (Y), \sigma_\varphi^- (Y)\}\}\}$
$= \min\{\max\{|p(A)| \lambda_\mu^-(X), |p(A)| \sigma_\varphi^-(X)\}, \max\{|p(A)| \lambda_\mu^-(Y), |p(A)| \sigma_\varphi^-(Y)\}\}$
$= \min\{\max\{((A\lambda_\mu)^P)^- (X), ((A\sigma_\varphi)^P)^- (X)\}, \max\{((A\lambda_\mu)^P)^- (Y), ((A\sigma_\varphi)^P)^- (Y)\}\}$
$= \min\{((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^- (X), ((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^- (Y)\}\}$

Hence the intersection of any two pseudo bipolar anti-fuzzy HX subgroups of a HX group ϑ is also a pseudo bipolar anti-fuzzy HX subgroup of ϑ.

4.17 Theorem

Union of any two pseudo bipolar anti-fuzzy HX subgroup of a HX group ϑ is also a pseudo bipolar anti-fuzzy HX subgroup of ϑ.
Proof: Let $(A\lambda_\mu)^P = (((A\lambda_\mu)^P)^+, ((A\lambda_\mu)^P)^-)$ and $(A\sigma_\varphi)^P = (((A\sigma_\varphi)^P)^+, ((A\sigma_\varphi)^P)^-)$ be any two pseudo bipolar anti-fuzzy HX subgroups of a HX group ϑ.

Now, Union of any two pseudo bipolar anti-fuzzy HX subgroup on a HX subgroup ϑ is

$((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P) = (((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^+, ((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^-)$

i. $((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^+ (XY^{-1}) = \max\{((A\lambda_\mu)^P)^+ (XY^{-1}), ((A\sigma_\varphi)^P)^+ (XY^{-1})\}$
$= \max\{ |p(A)| \lambda_\mu^+ (XY^{-1}), |p(A)| \sigma_\varphi^+ (XY^{-1})\}$
$= |p(A)| \max\{ \lambda_\mu^+ (XY^{-1}), \sigma_\varphi^+ (XY^{-1})\}$
$\leq |p(A)| \max\{\max\{ \lambda_\mu^+ (X), \lambda_\mu^+ (Y)\}, \max\{ \sigma_\varphi^+ (X), \sigma_\varphi^+ (Y)\}\}$
$= |p(A)| \max\{\max\{ \lambda_\mu^+ (X), \sigma_\varphi^+ (X)\}, \max\{ \lambda_\mu^+ (Y), \sigma_\varphi^+ (Y)\}\}$
$= \max\{|p(A)|\{\max\{\lambda_\mu^+ (X), \sigma_\varphi^+(X)\}, \max\{ \lambda_\mu^+ (Y), \sigma_\varphi^+ (Y)\}\}\}$
$= \max\{\max\{ |p(A)| \lambda_\mu^+(X), |p(A)| \sigma_\varphi^+(X)\}, \max\{ |p(A)| \lambda_\mu^+(Y), |p(A)| \sigma_\varphi^+(Y)\}\}$
$= \max\{ \max\{((A\lambda_\mu)^P)^+ (X), ((A\sigma_\varphi)^P)^+ (X)\}, \max\{((A\lambda_\mu)^P)^+ (Y), ((A\sigma_\varphi)^P)^+ (Y)\}\}$
$= \max\{((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^+ (X), ((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^+ (Y)\}\}$

ii. $((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^-(XY^{-1}) = \min\{((A\lambda_\mu)^P)^-(XY^{-1}),\ ((A\sigma_\varphi)^P)^-(XY^{-1})\}$

$= \min\{\,|p(A)|\ \lambda_\mu^-(XY^{-1}),\ |p(A)|\ \sigma_\varphi^-(XY^{-1})\}$

$= |p(A)|\ \min\{\lambda_\mu^-(XY^{-1}),\ \sigma_\varphi^-(XY^{-1})\}$

$\geq |p(A)| \min\{\min\{\lambda_\mu^-(X), \lambda_\mu^-(Y)\}, \min\{\sigma_\varphi^-(X), \sigma_\varphi^-(Y)\}\}$

$= |p(A)| \min\{\min\{\lambda_\mu^-(X), \sigma_\varphi^-(X)\}, \min\{\lambda_\mu^-(Y), \sigma_\varphi^-(Y)\}\}$

$= \min\{|p(A)|\{\min\{\lambda_\mu^-(X), \sigma_\varphi^-(X)\}, \min\{\lambda_\mu^-(Y), \sigma_\varphi^-(Y)\}\}\}$

$= \min\{\min\{|p(A)|\ \lambda_\mu^-(X),\ |p(A)|\ \sigma_\varphi^-(X)\}, \min\{|p(A)|\ \lambda_\mu^-(Y),\ |p(A)|\ \sigma_\varphi^-(Y)\}\}$

$= \min\{\min\{((A\lambda_\mu)^P)^-(X), ((A\sigma_\varphi)^P)^-(X)\}, \min\{((A\lambda_\mu)^P)^-(Y), ((A\sigma_\varphi)^P)^-(Y)\}\}$

$= \min\{((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^-(X),\ ((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^-(Y)\}$

Hence the Union of any two pseudo bipolar anti-fuzzy HX subgroups of a HX group ϑ is also a pseudo bipolar anti-fuzzy HX subgroup of ϑ.

4.18 Theorem

Union of any two pseudo bipolar fuzzy HX subgroup of a HX group ϑ is also a pseudo bipolar fuzzy HX subgroup of ϑ.

Proof: Let $(A\lambda_\mu)^P = (((A\lambda_\mu)^P)^+, ((A\lambda_\mu)^P)^-)$ and $(A\sigma_\varphi)^P = (((A\sigma_\varphi)^P)^+, ((A\sigma_\varphi)^P)^-)$ be any two pseudo bipolar fuzzy HX subgroups of a HX group ϑ.

Now, Union of any two pseudo bipolar fuzzy HX subgroup on a HX subgroup ϑ is

$((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P) = (((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^+, ((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^-)$

i. $((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^+(XY^{-1}) = \max\{((A\lambda_\mu)^P)^+(XY^{-1}),\ ((A\sigma_\varphi)^P)^+(XY^{-1})\}$

$= \max\{\,|p(A)|\ \lambda_\mu^+(XY^{-1}),\ |p(A)|\ \sigma_\varphi^+(XY^{-1})\}$

$= |p(A)| \max\{\lambda_\mu^+(XY^{-1}),\ \sigma_\varphi^+(XY^{-1})\}$

$\geq |p(A)| \max\{\min\{\lambda_\mu^+(X), \lambda_\mu^+(Y)\}, \min\{\sigma_\varphi^+(X), \sigma_\varphi^+(Y)\}\}$

$= |p(A)| \min\{\max\{\lambda_\mu^+(X), \sigma_\varphi^+(X)\}, \max\{\lambda_\mu^+(Y), \sigma_\varphi^+(Y)\}\}$

$= \min\{|p(A)|\{\max\{\lambda_\mu^+(X), \sigma_\varphi^+(X)\}, \max\{\lambda_\mu^+(Y), \sigma_\varphi^+(Y)\}\}\}$

$= \min\{\max\{|p(A)|\ \lambda_\mu^+(X),\ |p(A)|\ \sigma_\varphi^+(X)\}, \max\{|p(A)|\ \lambda_\mu^+(Y),\ |p(A)|\ \sigma_\varphi^+(Y)\}\}$

$= \min\{\max\{((A\lambda_\mu)^P)^+(X), ((A\sigma_\varphi)^P)^+(X)\}, \max\{((A\lambda_\mu)^P)^+(Y), ((A\sigma_\varphi)^P)^+(Y)\}\}$

$= \min\{((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^+(X),\ ((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^+(Y)\}$

ii. $((A\lambda_\mu)^P \cap (A\sigma_\varphi)^P)^-(XY^{-1}) = \min\{((A\lambda_\mu)^P)^-(XY^{-1}),\ ((A\sigma_\varphi)^P)^-(XY^{-1})\}$

$= \min\{\,|p(A)|\ \lambda_\mu^-(XY^{-1}),\ |p(A)|\ \sigma_\varphi^-(XY^{-1})\}$

$= |p(A)|\ \min\{\lambda_\mu^-(XY^{-1}),\ \sigma_\varphi^-(XY^{-1})\}$

$\leq |p(A)|\ \min\{\max\{\lambda_\mu^-(X), \lambda_\mu^-(Y)\}, \max\{\sigma_\varphi^-(X), \sigma_\varphi^-(Y)\}\}$

$= |p(A)| \max\{\min\{\lambda_\mu^-(X), \sigma_\varphi^-(X)\}, \min\{\lambda_\mu^-(Y), \sigma_\varphi^-(Y)\}\}$

$= \max\{|p(A)|\{\min\{\lambda_\mu^-(X), \sigma_\varphi^-(X)\}, \min\{\lambda_\mu^-(Y), \sigma_\varphi^-(Y)\}\}\}$

$= \max\{\min\{|p(A)|\ \lambda_\mu^-(X),\ |p(A)|\ \sigma_\varphi^-(X)\}, \min\{|p(A)|\ \lambda_\mu^-(Y),\ |p(A)|\ \sigma_\varphi^-(Y)\}\}$

$= \max\{\min\{((A\lambda_\mu)^P)^-(X), ((A\sigma_\varphi)^P)^-(X)\}, \min\{((A\lambda_\mu)^P)^-(Y), ((A\sigma_\varphi)^P)^-(Y)\}\}$

$= \max\{((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^-(X),\ ((A\lambda_\mu)^P \cup (A\sigma_\varphi)^P)^-(Y)\}$

Hence the Union of any two pseudo bipolar fuzzy HX subgroups of a HX group ϑ is also a pseudo bipolar fuzzy HX subgroup of ϑ.

CONCLUSIONS

We have given the notion of the pseudo bipolar fuzzy cosets and pseudo bipolar fuzzy cosets of bipolar fuzzy HX subgroup and bipolar anti-fuzzy HX subgroup of a HX group. The union and intersection of pseudo bipolar fuzzy HX subgroups and pseudo bipolar anti-fuzzy HX subgroups of a HX group are discussed. We hope that our results can also be extended to other algebraic system.

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