ON THE SOLUTION OF PERTURBED QUANTUM HARMONIC OSCILLATOR: A WHITE NOISEANALYSIS

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ABSTRACT

This study presents the solutions on the oscillator's propagator in two-dimensional homogeneous electric field within the framework of white noise functional approach, by calculating the propagator, and extracting the eigenstates and eigenfunctions of the system. From the obtained propagator, wave functions and energy spectrum were extracted. Two-dimensional harmonic oscillator is given an external potential, subject to a two-dimensional electric field. The system is first solved by completing the square, which reduces the Hamiltonian from quadratic plus linear term to just quadratic and linear term. Wavefunctions and energy spectra of the perturbed harmonic oscillators were extracted. Wavefunctions and probability densities were obtained in the ground and first two excited states. It was shown that the electric field's influence on the harmonic oscillator simply translates the positions and the corresponding oscillator energies is being reduced.

KEYWORDS

Energy spectrum, Harmonic oscillator, Wavefunction, White noise, Path integral, Probability density

1. INTRODUCTION

Quantum mechanics gives a range of probabilities for where a particle might be rather than precise values for the position or momentum of a specific particle in a given region during a given period. Additionally, it discussed wave-particle duality, which states that the characteristics of a particle can be stated as a wave. As a result, one can think of its quantum state as a wave, and wave functions can change over time. Thus, the measuring process, which is among the most challenging components of understanding quantum systems, is where the probabilistic nature of quantum mechanics originates.

By creating a path integral formulation in 1948, Feynman integrated an approach to figuring out a particle's ultimate states in a quantum system [1]. This integral approach includes for all conceivable system histories between the beginning and end states. Calculating the propagator, which is represented mathematically as follows,

$$(Kx_1, x_2; t_2) = \mathcal{N} \int exp\left(\frac{i}{\hbar}\right) S[x] D[x] \tag{1}$$

is a popular use of the integral where the normalization factor is given by *N*. The propagator which is an analytical tool determines the probability that a particle will move from one place to another in a given amount of time or with a given quantity of energy and momentum.

The quantum mechanical harmonic oscillator is fundamental in many fields of physics, serving as a model for a wider range of systems, including electrodynamical field modes, molecular and solid vibration systems. This study aims to use white noise functional approach to solve the propagator of a perturbed harmonic oscillator subject to a homogeneous electric field, *V(x,y)=qE(x,y)*. Some applications of the white noise path integral approach to physical and quantum mechanical systems are reported here [2-6].

2. MATHEMATICAL PRELIMINARIES

This section presents the mathematical formulation in solving the perturbed quantum harmonic oscillator in the framework of white noise analysis.

2.1. The Feynman Path Integral

Hamiltonian's principle of least action states that motion of an object will definitely follow a path of least action and is represented by

$$S = \int_{t_1}^{t_2} L(x,\dot{x})d\tau \qquad (2)$$

Feynman recommended that the propagator in Equation (1) is interpreted as the sum of all particle's feasible paths from x_1 and x_2 where *D[x]* is the infinite-dimensional Lebesgue measure.

In its symmetric form, the propagator can be written in terms of the wave functions Ψ_n and energy spectrum E_n in this form

$$K(x_1,x_2;t_2) = \sum_n \varphi_n(r_2)\varphi_n^*(r_1)e^{-i(t''-t')E_n/\hbar}. \qquad (3)$$

The Lagrangian, denoted by the term L in Equation (2) is defined as follows:

$$L = \frac{1}{2}m\dot{x}^2 - V(x). \qquad (4)$$

Within the framework of white noise analysis, the paths are parameterized through white noise variables,

$$x(\tau) = y(\tau) + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} B(\tau). \qquad (5)$$

And the Feynman propagator in Equation (3) becomes

$$K(x_1,x_2;t_2) = \boldsymbol{E}\left\{I_0 exp\left[\frac{i}{\hbar}S(x)\right]\right\} \qquad (6)$$

where

$$I_0 = \mathcal{N}exp\left[\left(\frac{i+1}{2}\right)\int_0^t \omega(t)^2 d\tau\right]. \qquad (7)$$

2.2. Characteristic Functional

Characteristic Functional is the Fourier transform of the Gaussian measure $d\mu(\omega)$ and is given by

$$C(\xi) = \int E^* e^{i<\omega,\xi>}d\mu(\omega) = e^{-\frac{1}{2}|\xi|_0^2}, \xi \epsilon E. \qquad (8)$$

2.3. The S- and T- transforms

For a white noise functional spectrum, denoted by $\varphi(\omega)$, its S-Transform and T-Transform [7], can be written as

$$(S\varphi)(\xi) = C(\xi)\int \varphi(\omega)e^{\langle\omega,\xi\rangle}d\mu(\omega), \tag{9}$$

and

$$(T\varphi): \xi \, \varepsilon \, S \rightarrow (T\varphi)(\xi) = \int_{0^*}^{S} \varphi(\omega)e^{i\langle\omega,\xi\rangle}d\mu(\omega) \tag{10}$$

respectively, where the Gaussian measure $d\mu(\omega)$ is in charge of the fall-off. Moreover, S-Transform is also related to T-Transform in this manner,

$$(S\varphi)(\xi) = C(\xi)(T\varphi)(-i\xi), \tag{11}$$

and

$$(T\varphi)(\xi) = C(\xi)(S\varphi)(i\xi). \tag{12}$$

2.4. Donsker Delta Function

The function$\delta(B(t)-y)$, is known as the Donsker delta, which often materializes in calculating for the propagator, and it can alternatively be written as Fourier representation given by

$$\delta(B(t)-y) = \frac{1}{2\pi}\int \exp[i\lambda(\langle\omega, x_{[0,t]}\rangle - y)]d\lambda. \tag{13}$$

2.5. The Wave function of Quantum Harmonic Oscillator in a Homogeneous Electric field

The wave function equation for the quantum harmonic oscillator in a homogeneous electric field as the perturbing potential [8] is given by

$$\varphi_n(x) = \left(2^n n!\sqrt{\pi}\right)^{-\frac{1}{2}} H_n\left[\beta\left(x - \frac{qE}{m\Omega^2}\right)\right]\exp\left(-\frac{\beta^2}{2}\left(x-\frac{qE}{m\Omega^2}\right)^2\right) = \left(2^n n!\sqrt{\pi}\right)^{-\frac{1}{2}} H_n \left[\sqrt{\frac{m\Omega}{\hbar}}\left(x - \frac{qE}{m\Omega^2}\right)\right] \text{x} \exp\left[-\frac{m\Omega}{2\hbar}\left(x-\frac{qE}{m\Omega^2}\right)^2\right]. \tag{14}$$

The schematic flow of solving the propagator of quantum harmonic oscillator subject to a homogeneous two-dimensional electric field within the framework of white noise functional approach is shown in Figure 1.

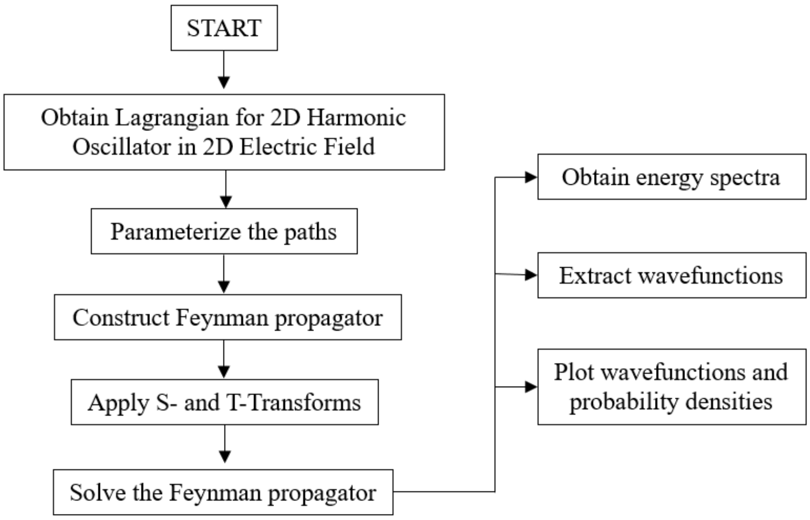

Figure 1. Schematic diagram for obtaining the propagator of a quantum harmonic oscillator subject to a perturbing potential.

3. Results and Discussions

This white noise functional approach in solving the perturbed quantum harmonic oscillator is presented in this section. Methods on calculating wavefunction, energy spectrum and wavefunction probability density for coupled harmonic oscillators in a two-dimensional electric field are presented.

3.1. Calculation of Quantum Harmonic Oscillator's Propagator in Two-Dimensions

The Lagrangian for two-dimensional harmonic oscillators in an electric field is given by

$$L(xy, \dot{x}\dot{y}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m\Omega^2(x^2 + y^2) + qE(x, y) \quad (15)$$

where

$$V_1(x, y) = \frac{1}{2}m\Omega^2(x^2 + y^2) \quad (16)$$

is the harmonic potential, Ω is the angular frequency and the potential due to a two-dimensional electric field is given by

$$V_2(x, y) = qE(x, y). \quad (17)$$

Parameterizing the path $\vec{X}(\tau)$ by considering Brownian $B(\tau)$ fluctuations about a sure path $\vec{Y}(\tau)$ led to the construction of the Feynman propagator of the form

$$K(0, X_2; t_1) = \boldsymbol{E}\left\{\begin{matrix} Nexp\left[\frac{(i+1)}{2}\langle\omega,\omega\rangle - \frac{i}{\hbar}\langle\omega, S''\omega\rangle + \frac{i}{\hbar}\frac{q^2E^2}{2m\Omega^2}\right] \\ \times\,\delta\left(B(\tau) - \left(\frac{m}{\hbar}\right)^{1/2} X_2\right) \end{matrix}\right\} \quad (18)$$

where the first term takes into account the relation between flat measure and Gaussian measure. The second term is due to harmonic oscillator's potential while the third is due to the uniform electric field potential perturbation. Moreover, applying the S- and T- transforms yields the Feynman propagator equation

$$K(0,x_2,t_2) = TI(\xi) = \left(\frac{m\Omega_{1,2}}{2\pi i\hbar sin\Omega_{1,2}t}\right)^{1/2} exp\left(\frac{i}{\hbar}\frac{q^2E^2}{2m\Omega^2}t + \frac{im\Omega_{1,2}X_2{}^2}{4\hbar tan\Omega_{1,2}t}\right) \quad (19)$$

$$K(0,x_2,t_2) = \frac{m}{2\pi i\hbar}\left(\frac{\Omega_1\Omega_2}{sin\Omega_1 t sin\Omega_2 t}\right)^{1/2} exp\left\{\left(\frac{i}{\hbar}\frac{q^2E^2}{2m\Omega^2}t\right) + \left[\frac{im\Omega_1}{4\hbar sin\Omega_1 t}cos\Omega_1 t\left(x_2 - \frac{qE}{m\Omega^2}\right)^2\right]\right\}$$
$$\times exp\left\{\frac{im\Omega_2}{4\hbar sin\Omega_2 t}\left[cos\Omega_2 t\left(x_2 + \frac{qE}{m\Omega^2}\right)^2\right]\right\} \quad . \quad (20)$$

3.2. Energy Spectrum and Wavefunctions of Two-Dimensional Harmonic Oscillators

Expanding the obtained propagator in terms of its eigenvalues and eigenfunctions gives

$$K(0,x_2,t_2) = \sum_{n_1,n_2\in N} e^{-\left(\frac{i}{\hbar}\right)E_{n1,n2}t}\, \varphi_{n1,n2}(0)\varphi_n{}^*(X_1,X_2) \quad (21)$$

for an initial point $X_2 = X_1 = 0$. Using the relations $isin\Omega t = \frac{1}{2}e^{i\Omega t}(1 - e^{-2i\Omega t})$and $cos\Omega t = \frac{1}{2}e^{i\Omega t}(1 + e^{-2i\Omega t})$, one can write the LHS of Equation (20) as

$$\left(\frac{m\Omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\left(\frac{i\Omega t}{2}\right)}(1 - e^{-2i\Omega t})^{-\frac{1}{2}} exp\left(\frac{i}{\hbar}\frac{q^2E^2}{2m\Omega^2}t\right) exp\left[-\frac{m\Omega}{2\hbar}X_2{}^2\frac{(1+e^{-2i\Omega t})}{(1-e^{-2i\Omega t})}\right] \quad (22)$$

where $X_2 = x_2 - \frac{qE}{m\Omega^2}$. To rewrite the propagator in its symmetric form, we use the Mehler's formula given by

$$\frac{1}{\sqrt{1-Z^2}} exp\left[\frac{4xyz-(x^2+y^2)(1-z^2)}{2(1+z^2)}\right] = e^{-\frac{(x^2+y^2)}{2}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{2}\right)^n H_n(x)H_n(y) \quad (23)$$

where functions H_n are the Hermite polynomials. Thus, the symmetric form of the propagator results to symmetric form of the propagator results to

$$K(0,X_2,t_2) = \frac{m}{\pi\hbar}\left[\frac{\Omega_1\Omega_2}{e^{it(\Omega_1+\Omega_2)}}\right]^{\frac{1}{2}} e^{\left(-\frac{i\Omega t}{2\hbar}\right)} e^{\left(\frac{i}{\hbar}\frac{q^2E^2}{2m\Omega^2}t\right)} \times exp\left[\begin{matrix}-\frac{m\Omega_1}{4\hbar}\left(x - \frac{qE}{m\Omega^2}\right)^2 \sum_{n_1=0}^{\infty}\frac{1}{n_1!}\left(\frac{e^{-i\Omega_1 t}}{2}\right)^{n_1} H_{n_1} \\ \left[\sqrt{\frac{m\Omega_1}{2\hbar}}\left(x - \frac{qE}{m\Omega^2}\right)\right]\end{matrix}\right]$$
$$\times exp\left[\begin{matrix}-\frac{m\Omega_2}{4\hbar}\left(y + \frac{qE}{m\Omega^2}\right)^2 \sum_{n_2=0}^{\infty}\frac{1}{n_2!}\left(\frac{e^{-i\Omega_2 t}}{2}\right)^{n_2} H_{n_2} \\ \left[\sqrt{\frac{m\Omega_2}{2\hbar}}\left(y + \frac{qE}{m\Omega^2}\right)\right]\end{matrix}\right] \quad (24)$$

$$K(0,X_2,t_2) = \frac{m}{\pi\hbar}(\Omega_1\Omega_2)^{\frac{1}{2}} exp\left\{-\frac{m}{4\hbar}\left[\Omega_1\left(x - \frac{qE}{m\Omega^2}\right)^2 + \Omega_2\left(y + \frac{qE}{m\Omega^2}\right)^2\right]\right\}$$
$$\times \sum_{n_1,n_2\in N}(2^{n_1}n_1!)^{-1}(2^{n_2}n_2!)^{-1} exp\left\{-it\left[\frac{1}{2}(\Omega_1+\Omega_2) + n_1\Omega_1 + n_2\Omega_2 - \left(\frac{q^2E^2}{2m\Omega^2}\right)\right]\right\}$$
$$\times H_{n_1}\left[\sqrt{\frac{m\Omega_1}{2\hbar}}\left(x - \frac{qE}{m\Omega^2}\right)\right] H_{n_2}\left[\sqrt{\frac{m\Omega_2}{2\hbar}}\left(y + \frac{qE}{m\Omega^2}\right)\right]. \quad (25)$$

One can immediately extract the energy spectrum, given by

$$E_{n_1,n_2} = \hbar\left[\left(\frac{1}{2} + n_1\right)\Omega_1 + \left(\frac{1}{2} + n_2\right)\Omega_2\right] - \frac{q^2E^2}{2m\Omega^2}. \quad (26)$$

And the general formula for the wavefunctions of quadratic harmonic oscillator subject to a perturbing potential, i.e., homogeneous electric field is of the form

$$\varphi_{n_1,n_2} = \sqrt{\frac{m\sqrt{\Omega_1\Omega_2}}{2^{(n_1+n_2)}\hbar\pi n_1!n_2!}} exp\left\{-\frac{m}{4\hbar}\left[\Omega_1\left(x - \frac{qE}{m\Omega^2}\right)^2 + \Omega_2\left(y + \frac{qE}{m\Omega^2}\right)^2\right]\right\}. \quad (27)$$

Thus, we have solved the K-propagator of two-dimensional oscillator in a homogeneous electric field using the white noise functional integral formulation. We have obtained the wavefunctions and energy spectra and have shown that the energy spectra are just the sum of the energies of two harmonic oscillators plus the energy due to the electric field where the system is subjected to. In addition, the quantum state of two-dimensional harmonic oscillators is described completely by its wavefunctions. These results agree with the works in [9-10]. Here, it was shown that the mathematically well-established approach, the white noise analysis approach can be utilized in solving the problem of perturbed quantum harmonic oscillator.

3.3. Plots of Oscillator's Wavefunctions and Probability Densities

3.3.1. Ground State

In terms of equivalent Hermite polynomial for $n_1 = n_2 = 0$, the respective wavefunction of two-dimensional perturbed quantum harmonic oscillator for the ground state is of the form

$$\Psi_{0,0} = \sqrt{\frac{m\sqrt{\Omega_1\Omega_2}}{4\pi\hbar}} exp\left\{-\frac{m}{4\hbar}\left[\Omega_1\left(x - \frac{qE}{m\Omega^2}\right)^2 + \Omega_2\left(y + \frac{qE}{m\Omega^2}\right)^2\right]\right\}. \qquad (28)$$

The ground state wavefunction and the probability density plots are shown in Figure 2 and Figure 3, respectively.

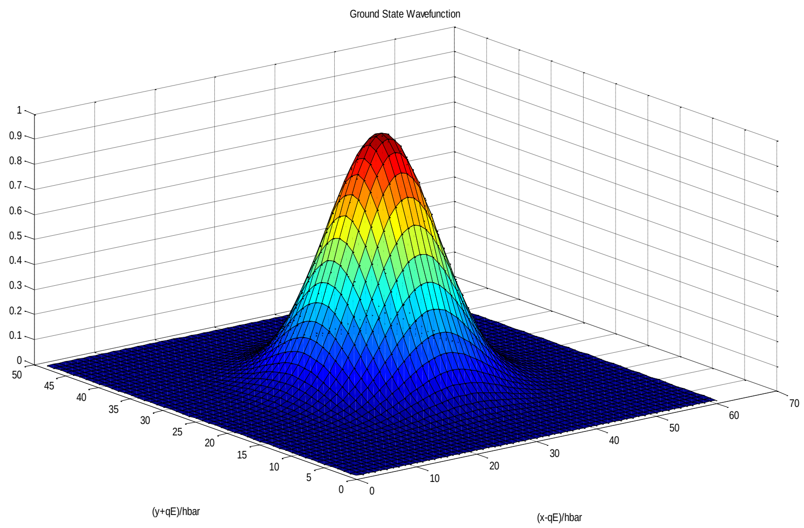

Figure 2. Ground state wavefunction, $\Psi_{0,0}$.

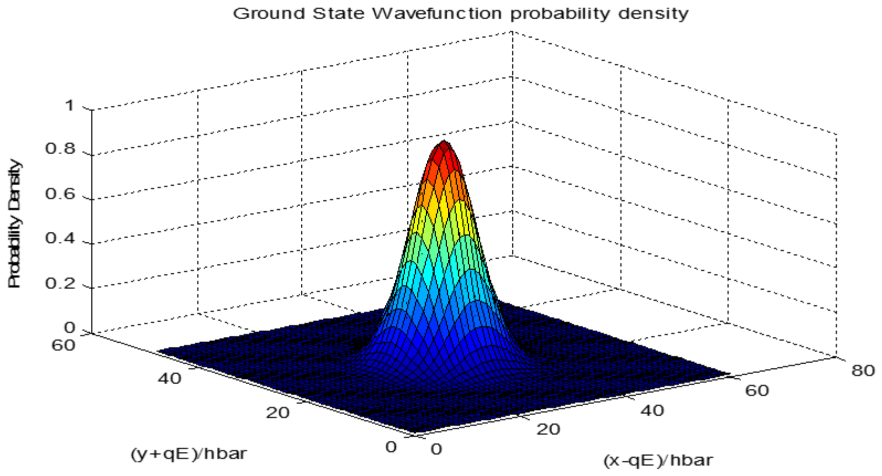

Figure 3. Ground State Wavefunction Probability Density, $|\Psi_{0,0}|^2, P_{0,0} = \langle\Psi_{0,0}|\Psi_{0,0}\rangle$.

3.3.2. First Excited State

The corresponding wavefunction for the first excited state $n_1 = n_2 = 1$, is of the form

$$\Psi_{1,1} = \sqrt{\frac{m\sqrt{\Omega_1\Omega_2}}{4\pi\hbar}} exp\left\{-\frac{m}{4\hbar}\left[\Omega_1\left(x - \frac{qE}{m\Omega^2}\right)^2 + \Omega_2\left(y + \frac{qE}{m\Omega^2}\right)^2\right]\right\} \left[\frac{\sqrt{m\Omega_1}}{2\hbar}\left(x - \frac{qE}{m\Omega^2}\right)\right] \times \left[\frac{\sqrt{m\Omega_1}}{2\hbar}\left(y + \frac{qE}{m\Omega^2}\right)\right]. \quad (29)$$

Figure 4 and Figure 5 show the wavefunction and the corresponding probability density for the first excited state, respectively.

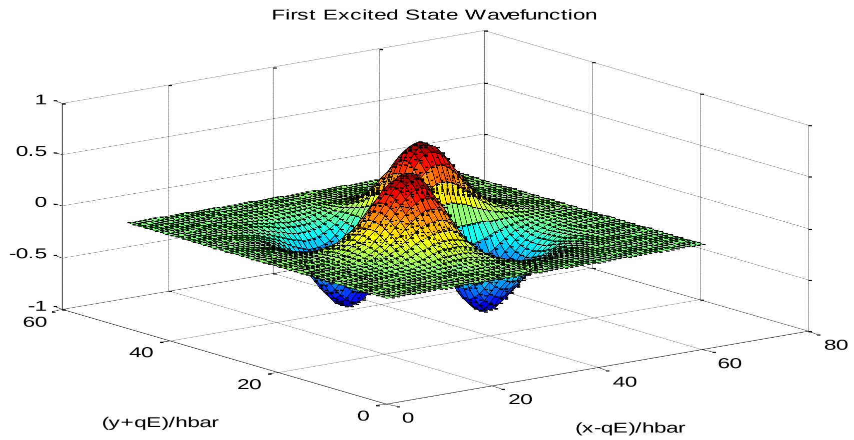

Figure 4. First excited state wavefunction, $\Psi_{1,1}$.

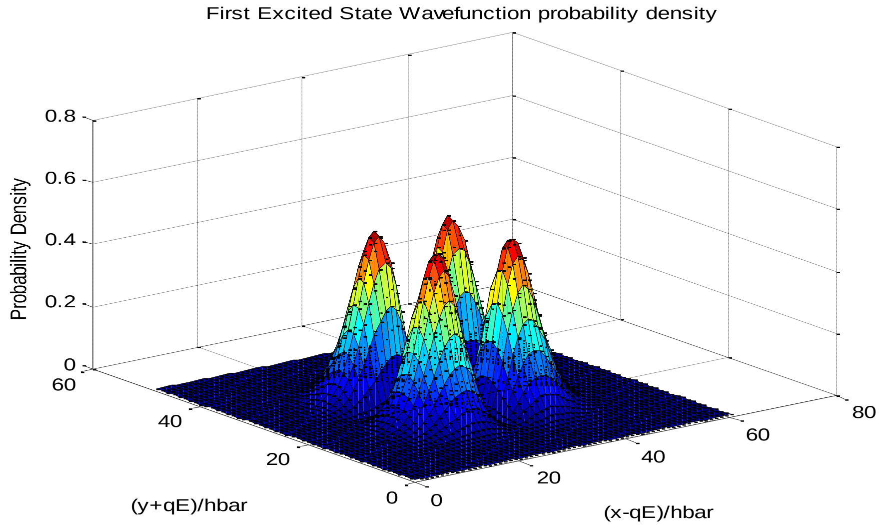

Figure 5. First Excited State Wavefunction Probability Density, $|\Psi_{1,1}|^2, P_{1,1} = \langle\Psi_{1,1}|\Psi_{1,1}\rangle$.

3.3.3. Second Excited State

At the second excited state, i.e., $n_1 = n_2 = 2$, the corresponding wavefunction is of the form

$$\Psi_{2,2} = \sqrt{\frac{m\sqrt{\Omega_1\Omega_2}}{64\pi\hbar}}exp\left\{-\frac{m}{4\hbar}\left[\Omega_1\left(x - \frac{qE}{m\Omega^2}\right)^2 + \Omega_2\left(y + \frac{qE}{m\Omega^2}\right)^2\right]\right\}$$
$$\times\left[2\left(\frac{\sqrt{m\Omega_1}}{2\hbar}\left(x - \frac{qE}{m\Omega^2}\right)\right)^2 - 2\right] \times \left[2\left(\frac{\sqrt{m\Omega_1}}{2\hbar}\left(x - \frac{qE}{m\Omega^2}\right)\right)^2 - 2\right]. \quad (30)$$

Figure 6 and Figure 7 show the plot of the wavefunction and its probability density for the second excited state, respectively.

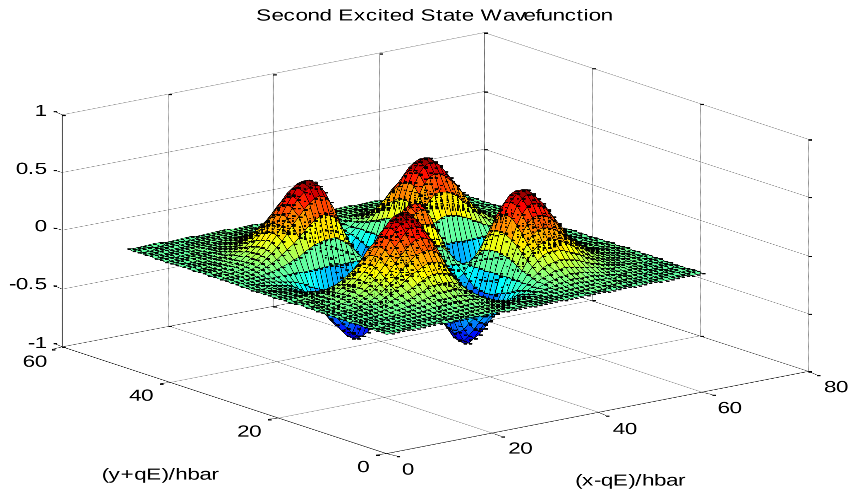

Figure 6. Second excited state wvefunction, $\Psi_{2,2}$.

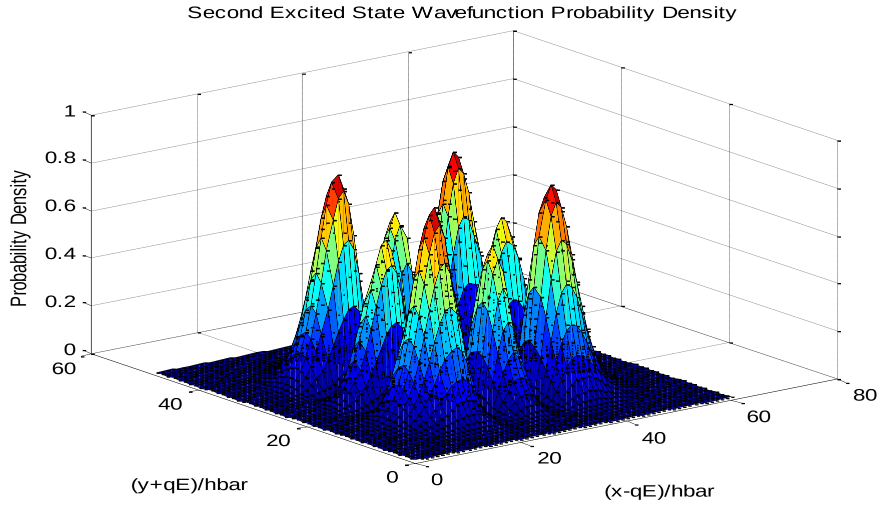

Figure 7. Second Excited Stae Waefunction Probability Density, $|\Psi_{2,2}|^2, P_{2,2} = \langle\Psi_{2,2}|\Psi_{2,2}\rangle$.

The obtained wavefunctions from this study are in agreement to the basic quantum mechanics where the oscillator is made to propagate along a two-dimensional space with the enclosed of the two-dimensional perturbing potential. The wavefunction amplitude as can be seen in the Figures 2-8 has different peaks that corresponds to the quantization of the open quantum system, including the two-dimensional electric field, the system's trajectory was shifted as expected from the previous assumption.

4. Conclusions

Using a white noise functional technique, we are able to determine the propagator of two-dimensional harmonic oscillators in a two-dimensional electric field in this study. Having the propagator solved, we extracted wavefunctions and energy spectrum for the system. And through equivalent Hermite polynomials, we obtained a simplified wavefunctions and probability densities of the harmonic oscillators for the first three quantum states. It can be noticed that the electric field's potential perturbation to the wavefunctions shifted the positions x into $x - \frac{qE}{m\Omega_1^2}$and y into $y + \frac{qE}{m\Omega_2^2}$. In addition, the energy spectrum obtained reduced the simple harmonic oscillator energies by a quantity$\frac{q^2E^2}{m\Omega^2}$. It has been shown that the white noise functional approach can be applied to quantum mechanical problems or systems such as the harmonic oscillator in a homogeneous electric field, coupled harmonic oscillators and others.

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REFERENCES

[1] Feynman, R & Hibbs, A, (1965) *Quantum Mechanics and Path Integrals,* McGraw-Hill, New York.
[2] Streit, L & Hida, T, (1983) "Generalized Brownian functionals and the Feynman integral", *Stoch Proc Appl.* Vol. 16, pp55-69.
[3] Bernido, CC & Carpio-Bernido, MV, (2014) *Methods and Applications of White Noise Analysis in Interdisciplinary Sciences*, World Scientific.
[4] Rangaig, NA, Peñonal GFI, Convicto, VC & Aringa, HP, (2017) "Quantum dynamic propagator of harmonic oscillator with uniform electric field in N-multimode harmonic oscillator bath: a white noise approach", Int J Math Anal. Vol. 11, No. 23, pp1103-1115.
[5] Rangaig, NA, Pada-Dulpina, CT & Convicto, VC, (2019), "Derivation of quantum propagator for coupled harmonic oscillator with uniform electric field in a single oscillator environment using white noise functional approach", *J King Saud Univ Sci.* Vol. 31, No. 2, pp252-256.
[6] Butanas, BM, (2020), "Dynamics of coupled harmonic oscillators in an environment using white noise analysis", *AIP Conf Proc.,* Vol. 2286, No. 1, 040002.
[7] Silva, JL & Streit, L, (2002), "Feynman integrals and white noise analysis", World Sci Pub., River Edge, NJ.
[8] Somerado, SAM & Convicto, VC (2017), *Analytical Solution of Perturbed Two-Dimensional Harmonic Oscillator Using White Noise Path Integral Approach*, Undergraduate Thesis, Mindanao State University, Marawi City.
[9] Lange, OL & Raab, RE, (1991), *Operator Methods in Quantum Mechanics,* Oxford University Press, New York.
[10] Lindao, ND (2007), *Harmonic Oscillator in a Uniform Electric Field: A White noise functional approach,* Graduate Thesis, Mindanao State University-Iligan Institute of Technology, Iligan City.

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